Volume 2 Number 7

Electronic Journal of Theoretical Physics

ISSN 1729-5254

EJTP

Editors

http://www.ejtp.com Augest, 2005 E-mail:info@ejtp.com

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Electronic Journal of Theoretical Physics

ISSN 1729-5254

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EJTP **7** (2005) 1–10 *Electronic Journal of Theoretical Physics*

Application of Coadjoint Orbits in the Thermodynamics of Non-Compact Manifolds.

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Received 20 May 2005, Published 20 Aug 2005

Abstract: Method of the solution of the main problem of homogeneous spaces thermodynamics for non-compact spaces in the case of non-compact Lie groups is presented in the article. The method is based on the method of coadjoint orbits. The formula that allows efficiently evaluate heat kernel on non-compact spaces is obtained. The method is illustrated by non-trivial example.

Keywords: Lie group; coadjoint representation; Darboux coordinates; harmonic analysis; homogeneous spaces thermodynamics; λ-representation; quantum equations; heat kernel; distribution function.
PACS (2003): 02.20.Rt; 02.30.Jr; 03.65.Fd
MSC2000: 17B63; 17C90; 22E70; 35Q05; 35Q40; 37K05; 37K40; 70E17; 70G65; 70H06; 81R12.

1. Introduction

The main goal of the thermodynamics of non-compact spaces is evaluation of the heat kernel Z_β as a sum over the spectrum of energy operator $H = -\Delta$ (Laplace-Beltrami operator)

$$Z_\beta = \sum_n d_n \exp(-\beta E_n), \tag{1}$$

where d_n is a degeneration of eigenvalue E_n, β - inverse temperature. In case of elliptic operator H on compact space, series (1) is always convergent [1].

Heat kernel (1) can be expressed as a trace of density matrix $\rho_\beta(x, x')$

$$Z_\beta = \int \rho_\beta(x,x)d\mu(x), \quad d\mu(x) = \sqrt{|g|}dx, \tag{2}$$

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which is a solution of Bloch equation

$$\frac{\partial \rho_\beta(x,x')}{\partial \beta} + H(x)\rho_\beta(x,x') = 0, \quad \rho_\beta(x,x')|_{\beta=0} = \delta(x-x')/\sqrt{|g|}. \tag{3}$$

Hence H is invariant operator, that problem is closely connected with the theory of Fourier analysis on non-compact spaces.

Many papers ([2],[3],[4]) were dedicated to the investigation of the properties of Laplace operator and heat equation, especially to high-temperature approximation of density matrix and a lot of important results were obtained there. For instance in case of n-dimensional compact space for Z_β is expresed as series

$$Z_\beta = \frac{Vol_M}{(4\pi\beta)^{n/2}} \sum a_i \beta^i, \tag{4}$$

were coefficients a_i represent spectral invariants, which may be expressed through functions on the manifold, symbol of operator H and it's derivatives.

The fact is that mathematical physics usually considers thermodynamics of compact manifolds or at least non-compact manifolds of finite volume. There is no general algorithm to factorize the volume of the manifold in sum (1) or in integral (2) in order to consider specific (by volume) heat kernel.

That work dealt with thermodynamics of non-compact manifolds (non-compact Lie groups as an example). The method of heat kernel evaluation is suggested here and it's based on the theory of coadjoint orbits. The method allows to factorize divergent volume of manifold so we can easily work with sufficiently finite specific (by volume) heat kernel.

Equation of type (3) are usually to be solved in the framework of the method of separation of variables, which allows to build the basis of solutions. Here we face a drawback connected with the problem of composition of the solution to satisfy stated initial conditions.

We use here method of orbits that efficiently uses the symmetries of the equation and we can remedy the drawback referring initial conditions mentioned above. Also the method allows to find global solutions, avoiding the problem of solution sewing in different coordinate maps even in case when the space cannot be covered with one map. The problem in the framework of the method of K-orbits is driven to the space of orbits with geometry and topology much simpler then the initial space had.

As an example of efficient application of the orbits method to problems of the thermodynamics of non-compact manifolds we thoroughly consider these problems for one non-compact unimodular Lie group. The case of the unimodular Lie groups was chosen as an illustrative example of presented method to avoid non-sufficient technical details which may make the main problem hard to understand.

2. Quantization of transition functions to the canonic coordinates on K-orbits

We introduce below basic constructions necessary for the solution of the problem, stated in the Introduction.

Let G be a real connected n-dimensional Lie group and $\mathcal{G}$ its Lie algebra. Group G acts on coalgebra $\mathcal{G}^*$ with coadjoint representation $\mathrm{Ad}^* : G \times \mathcal{G}^* \to \mathcal{G}^*,\ (g, f) \to \mathrm{Ad}^*_g\ f$ by convention:

$$\langle \mathrm{Ad}^*_g\ f, X\rangle \equiv \langle f, \mathrm{Ad}_{g^{-1}}\ X\rangle; \quad f \in \mathcal{G}^*,\ g \in G,\ X \in \mathcal{G}. \tag{5}$$

Here Ad_g is the linear operator of adjoint representation of group G on Lie algebra $\mathcal{G}$. Coordinates of covector f in the basis dual to Lie algebra $\mathcal{G}$ basis $\{e_i\}$ are denoted as $f_i,\ f = f_i e^i,\ \langle e^i, e_j\rangle = \delta^i_j$. Hence the formula (5) in the coordinates looks as $(\mathrm{Ad}^*_g f)_i = (\mathrm{Ad}_{g^{-1}})^j_i f_j$.

The linear degenerate Poisson bracket is defined on the dual space $\mathcal{G}^*$:

$$\{\varphi, \psi\}(f) \equiv \langle f, [\nabla\varphi(f), \nabla\psi(f)]\rangle; \quad \varphi, \psi \in C^\infty(\mathcal{G}^*), \tag{6}$$

where $\nabla\varphi(f) \equiv e_i \partial\varphi(f)/\partial f_i$.

Let's denote as ω_λ a symplectic 2-form (Kirillov form) on an orbit $\mathcal{O}_\lambda$ which acts on vectors tangent to orbit as follows:

$$\omega_\lambda(a, b) = \langle \lambda, [\alpha, \beta]\rangle, \quad a = ad^*_\alpha\lambda \in T_\lambda\mathcal{O}_\lambda,\ b = ad^*_\beta\lambda \in T_\lambda\mathcal{O}_\lambda,\ \alpha, \beta \in \mathcal{G}. \tag{7}$$

It's obvious that the Poisson bracket determined by Kirillov form ω_λ on an orbit $\mathcal{O}_\lambda$ coincides with the restriction of the Poison bracket (6) to that orbit.

According to the Darboux theorem the local canonical coordinates (the Darboux coordinates) exist on an orbit $\mathcal{O}_\lambda$ in which the form ω_λ looks like

$$\omega_\lambda = \sum_{a=1}^{\frac{1}{2}\dim\mathcal{O}_\lambda} dp_a \wedge dq^a.$$

It's easy to see that the transition to the canonical Darboux coordinates $(f_i) \to (p_a, q^a)$ means to find a set of analytic functions of the variables (p, q): $f_i = f_i(q, p, \lambda)$ that satisfy following conditions

$$f_i(0, 0, \lambda) = \lambda_i; \tag{8}$$

$$\frac{\partial f_i(q,p,\lambda)}{\partial p_a}\frac{\partial f_j(q,p,\lambda)}{\partial q^a} - \frac{\partial f_j(q,p,\lambda)}{\partial p_a}\frac{\partial f_i(q,p,\lambda)}{\partial q^a} = C^k_{ij} f_k(q,p,\lambda); \tag{9}$$

$$K_\mu(f(q,p,\lambda)) = K_\mu(\lambda); \quad \mu = 1, \ldots, \operatorname{codim}\mathcal{O}_\lambda. \tag{10}$$

(Here $K_\mu(f)$ are the Casimir function for orbit $\mathcal{O}_\lambda$, i.e. such functions from the space $C^\infty(\mathcal{G}^*)$ that $\{\varphi(f), K_\mu(f)\}|_{\mathcal{O}_\lambda} = 0,\ \forall\varphi \in C^\infty(\mathcal{G}^*)$).

Let's require the transition to the canonical coordinates (in other words, the qp–transition) to be linear on p_a, i.e.

$$f_i(q,p,\lambda) = \alpha_i^a(q)p_a + \chi_i(q,\lambda); \quad \operatorname{rank}\alpha_i^a(q) = \frac{1}{2}\dim\mathcal{O}_\lambda. \tag{11}$$

In general, the linear transition (11) does not exist but if we assume that $\alpha_i^a(q)$ and $\chi_i(q;\lambda)$ are the holomorphic functions of the complex variables q then the class of a K-orbits and a Lie algebras which might allow the linear transition becomes much wider. (We assume that functionals from $\mathcal{G}^*$ are extended linearly on $\mathcal{G}^c$). The domain of definition Q of variables q and the domain of definition P of variables p is determined from the requirement for the function $f_i(q,p,\lambda)$ to be real-valued.

The following theorem is proven in [5].

Theorem 1. *Linear transition to the canonical coordinates on orbit $\mathcal{O}_\lambda$ exists only when for a linear functional λ the normal polarization does exist, i.e. the subalgebra $\mathcal{H}\subset\mathcal{G}^c$ such as*

$$\dim\mathcal{H} = n - \frac{1}{2}\dim\mathcal{O}_\lambda, \quad \langle\lambda,[\mathcal{H},\mathcal{H}]\rangle = 0, \quad \lambda+\mathcal{H}^\perp\subset\mathcal{O}_\lambda.$$

The linear transition (11) is useful in many aspects. Firstly, the procedure of the Poisson bracket quantization is easily obtained because in this case the problem of regularization of operators does not arise (see below). Secondly, quadratic hamiltonians maps (when the transition (11) applied) into also quadratic hamiltonians in respect to impulses and after the quantization we get second order equations. The linear qp-transition exists in the overwhelming majority of cases because it's known (see, e.g. [6]) that for an arbitrary Lie algebra and for any non-degenerate covector a solvable polarization does exist.

We introduce notation j_μ ($\mu = 1,\ldots,\operatorname{ind}(\mathcal{G})$) for local coordinates in the space of orbits $\mathcal{G}^*/G$. We fix representative of an orbit $\lambda = \lambda(j)$, assuming that λ depends on j linearly. Further we will consider covector λ to be non-degenerate, i.e. $\dim\mathcal{O}_\lambda$ to have maximal value.

Let's define the concept of the K-orbit quantization, see [5]. The quantization must be done separately for each type of orbits. It consists in the correspondence to a type of orbits the special infinite-dimensional Lie algebra representation, so called (λ-*representation*). The condition of *integrality* is imposed on the orbits. We remind [7], that an orbit called integer-valued if the Kirillov form integral taken over any 2-dimensional cycle is integer-valued.

Let's consider the transition functions $f_i(q,p;\lambda)$ as the symbols of operators. They are defined as follows, variables p_a are substituted with the differential operators $p_a \to \hat{p}_a \equiv -i\hbar\partial_{q^a}$. Then the covector's coordinates f_i transform into the linear operators $f_i(q,p;\lambda) \to \hat{f}_i = f_i(q,-i\hbar\partial_q;\lambda+i\hbar\beta))$ (here $\hbar$ is a real positive parameter). Under the quantization an arbitrary function $\varphi(f)\in C^\infty(\mathcal{G}^*)$ is assigned with a symmetrized operator function $\varphi(\hat{f})$ of the operators $\hat{f}_i$. The real vector β is determined from the real-value requirement for the functions:

$$\kappa_\mu(\lambda) = K_\mu(\hat{f}). \tag{12}$$

The procedure of quantization is ambiguous. This ambiguity vanishes if we require the operators $\hat{f}_i$ to satisfy the following commutation rules

$$\frac{i}{\hbar}[\hat{f}_i, \hat{f}_j] = C^k_{ij}\hat{f}_k. \tag{13}$$

It's obvious that if the transition to the canonical coordinates is linear, i.e. for this type of an orbit exists a normal polarization, then

$$\hat{f}_i = -i\hbar\alpha^a_i(q)\frac{\partial}{\partial q^a} + \chi_i(q, \lambda + i\hbar\beta)). \tag{14}$$

3. Quantum equations on K–orbits and heat kernel

We define here special infinite-dimensional irreducible representation of Lie algebra $\mathcal{G}$ which we shall call λ-representation. Operators of λ-representation $l_i(q, \partial_q, \lambda)$ are constructed as follows from the quantized canonic transition functions

$$l_i(q, \partial_q, \lambda) = \frac{i}{\hbar}\hat{f}_i(q, \partial_q, \lambda). \tag{15}$$

Let λ be a non-degenerate covector that belongs to an integer-valued orbit. Here following conventions are assumed, the group G is unimodular; $l_i(q, \partial_q, \lambda)$ is a λ-representation of Lie algebra $\mathcal{G}$; dg is an invariant measure on Lie group G; $d\mu(q)$ is the measure on Q such that operators $l_i(q, \partial_q, \lambda)$ are skew-symmetric with respect to the scalar product in $L_2(Q, d\mu(q))$. Note that the left-invariant vector fields (ξ_i) and the right-invariant vector fields (η_i) on G also are skew-symmetric with respect to the scalar product in $L_2(G, dg)$.

We introduce generalized function $D^\lambda_{qq'}(g)$ as the solution of the overdetermined system

$$[\xi_i(g) + l_i(q, \partial_q, \lambda)]D^\lambda_{qq'}(g) = 0; \;\; [\eta_i(g) + \bar{l}_i(q', \partial'_q, \lambda)]D^\lambda_{qq'}(g) = 0. \tag{16}$$

(since $D^\lambda_{qq'}(g)$ are the generalized functions then all of the equalities that include them (e.g. equation (16)) must be considered in the weak sense).The system (16) is consistent but global solutions exist only for integer-valued orbits. By definition functions $D^\lambda_{qq'}(g)$ appear to be the eigenfunctions of the Casimir operators and they are determined by the system (16) with an accuracy to a constant factor. This arbitrariness is eliminated by introduction of the normalization requirement $D^\lambda_{qq'}(e) = \delta(q, q')$, where $\delta(q, q')$ is the Delta-function with respect to the measure $d\mu(q)$. Note that the functions $D^\lambda_{qq'}(g)$ are to be found in quadratures because the system (16) is overdetermined.

Functions $D^\lambda_{qq'}(g)$ on unimodular Lie groups have following properties

$$\begin{aligned}
&\int D^j_{qq''}(x)D^j_{q''q'}(x')d\mu(q'') = D^j_{qq'}(x'x),\\
&D^j_{qq'}(x) = \overline{D}^j_{q'q}(x^{-1}),\\
&D^j_{qq'}(e) = \delta(q, q').
\end{aligned} \tag{17}$$

The set of functions $D^{\lambda}_{qq'}(g)$ is full and orthogonal [5] and since that satisfies following equations

$$\int\limits_{G} \overline{D}^{\tilde{j}}_{\tilde{q}\tilde{q}'}(g) D^{j}_{qq'}(g)\, dg = \delta(q,\tilde{q})\delta(q',\tilde{q}')\delta(j,\tilde{j}); \tag{18}$$

$$\int\limits_{Q\times Q\times J} \overline{D}^{j}_{qq'}(\tilde{g}) D^{j}_{qq'}(g)\, d\mu(j) d\mu(q) d\mu(q') = \delta(g,\tilde{g}). \tag{19}$$

Because of that for any function $\varphi(g)$ from the dense kernel subspace in $L_2(G,dg)$ the generalized Fourier transformation and inversion are defined:

$$\hat{\varphi}_{\lambda}(q,q') = \int \varphi(g) D^{\lambda}_{qq'}(g)\, dg \tag{20}$$

$$\varphi(g) = \int \hat{\varphi}_{\lambda}(q,q') \overline{D^{\lambda}_{qq'}(g)}\, d\mu(q) d\mu(q') d\mu(\lambda). \tag{21}$$

Here $d\mu(\lambda)$ is the spectral measure of Casimir operators $K_{\mu}(i\hbar\xi)$ $(\mu = 1, \ldots, r)$.

In the conventional method of harmonic analysis the functions that perform Fourier transformation are the eigenfunctions for a full n-dimensional set of commuting operators from an enveloping algebras of right- and left-invariant fields $U(\mathcal{G}^L), U(\mathcal{G}^R)$ [8]. The suggested approach has the significant advantage: if the functions $\varphi(g)$ and $\hat{\varphi}_{\lambda}(q,q')$ are connected by the transformations (20), (21) then the same transformation connects the action of the following operators

$$\xi_i(g)\varphi(g) \iff l_i(q,\partial_q,\lambda)\hat{\varphi}_{\lambda}(q,q'), \quad \eta_i(g)\varphi(g) \iff \bar{l}_i(q',\partial_{q'},\lambda)\hat{\varphi}_{\lambda}(q,q') \tag{22}$$

Let's put in correspondence to the function $H(f)$ on the coalgebra the symmetric function $H(i\hbar\eta)$ of operators $i\hbar\eta_i$ and consider the following differential equation in $L_2(G,dg)$:

$$H(i\hbar\eta)\varphi(g) = E\varphi(g). \tag{23}$$

The equation (23) is called *quantum equation on the Lie group* G [9].

Using transformation (21) and formula (22) we get the equivalent equation for the function $\hat{\varphi}_{\lambda}(q,q')$

$$H(\hat{f}(q,\lambda))\hat{\varphi}_{\lambda}(q,q') = E\hat{\varphi}_{\lambda}(q,q'). \tag{24}$$

Let's consider operator $H(x)$

$$H(x) = G^{ab}(i\hbar\eta_a)(i\hbar\eta_b), \tag{25}$$

where G^{ab} is arbitrary constant symmetric non-degenerate matrix of positively defined quadratic form, η_a — right-invariant vector fields on Lie group. It's easy to see that in this case operator H is Laplace operator on Lie group with rightinvariant metric

$$g^{ij}(x) = G^{ab}\eta^i_a(x)\eta^j_b(x). \tag{26}$$

Obviously Riemannian measure un Lie group $\sqrt{g}dx$ coincides (to the constant factor) with bi-invariant measure on Lie group.

Since set $D^{\lambda}_{qq'}(g)$ is full, the solution of equation (3) may be expressed as follows

$$\rho_\beta(x,x') = \int \tilde{\rho}_\beta(q,q',j,\tilde{q},\tilde{q}',\tilde{j}) D^{j}_{qq'}(x) \overline{D^{\tilde{j}}_{\tilde{q}\tilde{q}'}}(x') d\mu(q) d\mu(q') d\mu(j) d\mu(\tilde{q}) d\mu(\tilde{q}') d\mu(\tilde{j}), \quad (27)$$

where function $\tilde{\rho}_\beta(q,q',j,\tilde{q},\tilde{q}',\tilde{j})$ satisfies equation

$$\frac{\partial \tilde{\rho}_\beta(q,q',j,\tilde{q},\tilde{q}',\tilde{j})}{\partial \beta} + H(\hat{f}) \tilde{\rho}_\beta(q,q',j,\tilde{q},\tilde{q}',\tilde{j}) = 0, \quad (28)$$
$$\tilde{\rho}_\beta(q,q',j,\tilde{q},\tilde{q}',\tilde{j})|_{\beta=0} = \delta(q,\tilde{q})\delta(q',\tilde{q}')\delta(j,\tilde{j}).$$

Although $H(f)$ in (24) depends only on q, solution of (28) may me written down as

$$\tilde{\rho}_\beta(q,q',j,\tilde{q},\tilde{q}',\tilde{j}) = \mathcal{R}_\beta(q,\tilde{q},j)\delta(q',\tilde{q}')\delta(j,\tilde{j}) \quad (29)$$

Function $\mathcal{R}_\beta(q,\tilde{q},j)$ also satisfies equation

$$\frac{\partial \mathcal{R}_\beta(q,\tilde{q},j)}{\partial \beta} + H(\hat{f}) \mathcal{R}_\beta(q,\tilde{q},j) = 0, \quad \tilde{\mathcal{R}}_\beta(q,\tilde{q},j)|_{\beta=0} = \delta(q,\tilde{q}). \quad (30)$$

Let's substitute (30) in (28) and integrate over $\tilde{q}$ and $\tilde{j}$ taking into account special properties (28) of functions $D^{j}_{qq'}(x)$ for unimodular Lie groups, we will get final expression for density matrix on unimodular Lie group:

$$\rho_\beta(x,x') = \int \mathcal{R}_\beta(q,\tilde{q},j) D^{j}_{q\tilde{q}}(x'^{-1}x) d\mu(q) d\mu(\tilde{q}) d\mu(j). \quad (31)$$

One might see, that diagonal elements of density matrix have no dependence on group coordinates x. Heat kernel on Lie group as a trace of density matrix (31) is defined as follows

$$Z_\beta = \int \rho_\beta(x,x) d\mu(x) = \int d\mu(x) \int \tilde{\rho}_\beta(q,q,j) d\mu(q) d\mu(j) =$$
$$= Vol_G \int \tilde{\rho}_\beta(q,q,j) d\mu(q) d\mu(j), \quad (32)$$

where integration is performed over entire volume of the group.

One can easily see that specific (by volume) heat kernel (statistic sum) $z_\beta \equiv Z_\beta / Vol_G$ is sufficiently finite because operator H is elliptic.

As an example we will consider the problem of heat kernel on one non-compact Lie group.

4. Example

As an example of the method we will consider 3-dimensional Lie group $E(2)$. Lie algebra of that group is determined by following commutation rules:

$$[e_1, e_2] = 0, \quad [e_2, e_3] = \varepsilon e_1, \quad [e_1, e_3] = -\varepsilon e_2.$$

Lie group $E(2)$ is popular subject of study in theoretical physics and mathematics but usually parameter ε is taken equal 1. That case corresponds with Lie algebra of group of motions of two-dimensional euclidean plane. Other well known value of ε is 0 which represents three dimensional abelian Lie group. We take in consideration arbitrary parameter ε to be able to perform contraction $\varepsilon \to 0$.

Matrix $G^{ab} = \mathrm{diag}(A, B, C)$ and in given case we assume $\det G^{ab} > 0$.

Left-invariant and right-invariant vector fields on that group are

$$\xi_1 = \cos(\varepsilon x_3)\partial_1 + \sin(\varepsilon x_3)\partial_2, \quad \xi_2 = -\sin(\varepsilon x_3)\partial_1 + \cos(\varepsilon x_3)\partial_2, \quad \xi_3 = \partial_3,$$

$$\eta_1 = -\partial_1, \quad \eta_2 = -\partial_2, \quad \eta_3 = \varepsilon x_2\partial_1 - \varepsilon x_1\partial_2 - \partial_3.$$

According (25) we get operator $H(i\hbar\eta)$

$$H = -\hbar^2(A\eta_1^2 + B\eta_2^2 + C\eta_3^2).$$

Canonic transition functions are:

$$f_1 = -j\cos(\varepsilon q), \quad f_2 = j\sin(\varepsilon q), \quad f_3 = p.$$

Functions on Lagrange submanifold Q to the symplectic sheet, i.e functions of variable q here here are functions on S^1 which are $2\pi/\varepsilon$ periodic.

Set of functions $D^j_{qq'}$ we obtain as a solution of (16) and it satisfies (17), (18) and (19), looking as follows

$$D^j_{qq'} = \exp[\frac{ij}{\hbar}(-x_1\cos(\varepsilon q) + x_2\sin(\varepsilon q))]\delta(x_3 + q - q')$$

and we choose measure $d\mu(j) = \varepsilon j dj/(2\pi\hbar)^2$.

The case we are interested in is bolzmannian gas (gas of non-interacting particles of mass m). Because of that we have following restriction put on matrix G: $A = B = C = 1/2m$ and thus $H(\hat{f})$ is a hamiltonian of a free particle.

Operator $H(\hat{f})$, that is built according (24) from quantized canonic transition functions and enters (30):

$$H(\hat{f}) = \frac{1}{2m}(\hat{f}_1^2 + \hat{f}_2^2 + \hat{f}_3^2). \tag{33}$$

Since $\tilde{\mathcal{R}}_\beta(q, \tilde{q}, j)$ on Lagrange submanifold Q to the orbit $\mathcal{O}_\lambda$ must be $2\pi/\varepsilon$ periodic we must build a solution of the equation (24) as a series using Fourier transformation over variable q.

So we get density matrix $\mathcal{R}_\beta(q, \tilde{q}, j)$ on Lagrange submanifold Q to the orbit $\mathcal{O}_\lambda$, as a solution of (24) with operator (33):

$$\tilde{\mathcal{R}}_\beta(q, \tilde{q}, j) = \frac{\varepsilon}{2\pi}\sum_{n=-\infty}^{\infty}\exp(-\frac{1}{2m}((n\hbar\varepsilon)^2 + j^2)\beta - in\varepsilon(q - \tilde{q})) =$$

$$= \frac{\varepsilon}{2\pi}\exp(-\frac{j^2\beta}{2m})\theta_3(\frac{(q-\tilde{q})\varepsilon}{2}, \exp(-\frac{\beta\hbar^2\varepsilon^2}{2m})).$$

Specific statistic sum obtained from formula (32) is

$$z = (\frac{m}{2\pi\beta\hbar^2})^{3/2}\theta_3(0, \exp(-\frac{2m\pi^2}{\hbar^2\varepsilon^2\beta})) \tag{34}$$

which can be written down for more convenient consideration of thermodynamic properties as follows (θ_3 is θ_3 Jacobi function)

$$z = \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2} \theta_3(0, \exp(-\frac{T}{T_0(\varepsilon)})), \tag{35}$$

where k_B — Bolzmann constant and $T_0(\varepsilon) = \hbar^2\varepsilon^2/2mk_B\pi^2)$.

Mean energy of the particle in that space is

$$u = k_BT^2\frac{\partial \ln z}{\partial T}$$

and we can get specific heat (for one particle) in given space.

Asymptotic transition (contraction) $\varepsilon \to 0$ corresponds to the specific heat for the bolzmannian gas of free particles in flat space R^3.

Here we give the graph for the specific heat c_v of bolzmannian gas (for one particle) depending on $t = T/T_0(\varepsilon)$.

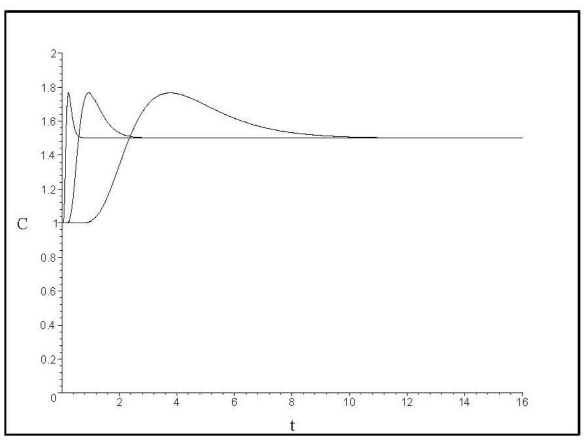

Fig. 1

On the graph one can see that lower temperatures cause elimination of one degree of freedom but at higher temperatures particles behave just like in three dimensional flat space. That abnormal behavior of the specific heat can be explained by the fact that particle "feels" topology of the space. Indeed the group space of $E(2)$ is a cylinder with non-trivial topology which strongly influences thermodynamical properties of bolzmannian gas at low temperatures.

The principal result of the paper is formula (32) that solves the main problem of thermodynamics of homogeneous spaces for non-compact unimodular Lie groups.

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The Boundary Conditions Geometry in Lattice-Ising Model

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Received 11 June 2005 , Published 20 August 2005

Abstract: We found that the differential topology of the lattice-system Ising model determines whether there can be the continuous phase transition, The geometric topology of the space the lattice-system is embedded in determines whether the system can become ordered. If the system becomes ordered it may not admit the continuous phase transition. The spin-projection orientations are strongly influenced by the geometric topology of the space the lattice system is embedded in.

Keywords: : Ising model, periodic boundary, spin-projection orientation, topology
PACS (2003): 64.60.Cn,05.50.+q,02.40.Pc

1. Introduction

Physicists worked on Ising models for many years have obtained brilliant achievements in the theory of continuous phase transition. The model's charm is that the studying and researching will never be at an end; some new properties and problems will be discovered and explored. Kadanoff pointed out, "The physical problem of critical phenomena reduces to the mathematical problem of enumerating and describing the different universality class."[1] The late Chinese mathematician Xing-sheng Chen, well-known for his great achievements in differential topology around the world, said, "Physics just is geometry, and they are in one family." In this paper we will discuss how the difference between differential and geometric topology influences the magnetic order of the lattice system of Ising model.

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2. Applications

It used to be that the discussion about the continuous phase transition of the Ising model concentrated on the property of derivatives of order 2 of the system's free energy at the critical temperature. If it is divergent the system will have magnetic order below the critical temperature, if it is not divergent the ordered state of system will not appear. If we regard the thermodynamic properties of the system as a kind of manifold, the space constructed by the system's thermodynamic parameters becomes a chart on the manifold, and the system's free energy is a function defined in the coordinate system.

The property of derivatives of order 2 of the free energy reflects the differential topology of the manifold. The phase transitions of the systems with the same differential topology belong to one kind of phase transition—a phase transition of the second kind. The order parameters are actually the invariance linked to the differential topology of such systems. Consequently, the discussion about the phase transition of the Ising model has a meaning in terms of differential topology, although people did not pay much attention to relate the phase-transition property to the property of differential topology, and were willing to consider it as a physical property only. As a result of this, they were so indifferent to the geometric topological property of the space in which the lattice system is embedded, that the selection of spin-projection orientations has never been mentioned, This aspect is dependent upon the geometric topology of the space. Considering the spin-projection orientations, the lattice gas will not be equivalent to the Ising model because the lattice gas itself has not any projection orientation [2].

The property of differential topology and the property of geometry topology are two distinct properties of topology: the former determines whether there can exist the continuous phase transition in a lattice system, and the later determines whether the system can become ordered. Being a continuous phase transition, the system will certainly become ordered, which was proved by physicists. But if the system behaves ordered, it may not be in the state of continuous phase transition for some systems. This results from the difference between the two topologies above mentioned. So it is necessary to discuss the influence of the geometric topology on the magnetic order of the Ising model. We will discuss the problems in the following three respects.

First, the geometric topology of the embedding space for the lattice system will restrict the spin-projection orientations. A key point of Onsager's solution is his periodic boundary conditions. Under the conditions, as illustrated in FIG .1, the two lattices in sites ($-\infty, y$) or (x,$-\infty$) and ($+\infty, y$) or (x,$+\infty$) have the same spin state. The periodic boundary conditions change the plane square-lattice system into a torus-lattice system, which should be embedded in 3-dimensional Euclidean space. If a lattice system can become ordered, it will be in one of the following two states: The entire lattice system can be referred to as a lattice with total spin. So, the space is simply connected; or there is a non-vanishing vector field, of which the appearance is dependent upon the geometric topology of the lattice system. The torus-lattice system is not contractible because of its

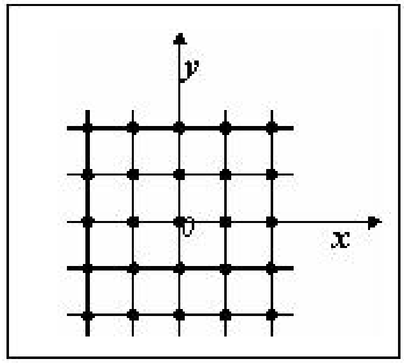

Fig. 1 Periodic boundary condition

geometric topological structure[3]. FIG .2 shows three simple closed curves on the torus, and only one's

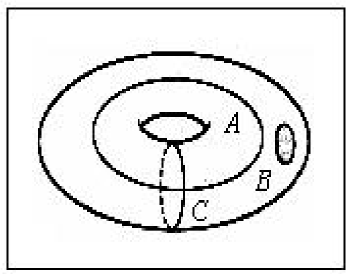

Fig. 2 Three simple closed curves on the torus

inner block of which, the curve B inner block, can become contractible. The singularity of Onsager's equation indicated that the system certainly has the continuous phase transition, which implied the system is ordered but contractible. We noticed that there are only two spin states for the Ising model: spin-up and spin-down, and there are only three important spin-projection orientations in various projecting orientations: the directions normal to the plane, parallel to the x-axis and to the y-axis as illustrated in FIG.1. In fact, the partition function of the system is not affected by the projecting orientations because there is no spin-projection orientation term in it. We found that the spin-projection orientation normal to the plane with the periodic boundary conditions is forbidden. The reason is that if the system were ordered, the total spin-projection orientation of the system should be normal to the torus surface, which cannot shrink. Then its projection orientation is uncertain because the normal orientations of the torus are different and divergent everywhere.

However, if the spin-projection orientations are parallel to the x-axis or to the y-axis, the situation is completely different. Let the total spin of the system be$S$ below its critical temperature, the average lattice spin be s, and $s = S/N$, with N the total number of the lattices in the system., as $N \to \infty$ and $s \neq 0$, in accord with the thermodynamic limit condition. When the torus-lattice system becomes ordered, there is a continuous non-vanishing vector field on the torus, as illustrated in FIG 3

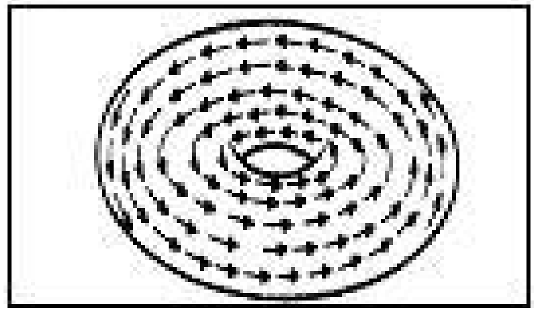

Fig. 3 A continuous non-vanishing vector field on the torus

each little arrow represents an average lattice spin, which means that the topological property of the torus allows the system to be ordered. It is the geometric topology that not only determines whether the system is ordered, but also restricts the spin-projection orientations. Onsager's solution means that the torus system's differential topology is equivalent to the plane system's differential topology, but the principle of topology tells us that their geometry topologies are different. Notwithstanding they cannot change into each other automatically,[i.e., they are not one-to-one].

In fact, the free energy of the torus-lattice system is different from the free- energy of the plane square-lattice system. The partition function of the latter is defined as

$$Q = \sum_{s_i} \exp(-\frac{H}{k_B T}) = \sum_{s_i}^{(1)} \exp(-\frac{H}{k_B T}) + \sum_{s_i}^{(2)} \exp(-\frac{H}{k_B T}) \tag{1}$$

$$H = -J \sum_{i,j} s_i s_j \tag{2}$$

where H is the system Hamiltonian, J is a spin-coupling constant, $\sum\limits_{i,j}$ denotes the sum over all possible nearest neighbor lattices, $\sum\limits_{s_i}$ does the sum over all possible states of spins, $\sum\limits_{s_i}^{(1)}$ is the sum over all possible states of spins with the periodic boundary conditions, and $\sum\limits_{s_i}^{(2)}$denotes the sum over all possible states of spins which fail to keep the conditions. The first term on the right hand of Eq.(1) is expressed by

$$Q_1 = \sum_{s_i}^{1} \exp(-\frac{H}{k_B T}) \tag{3}$$

obviously, Q_1 is the partition function of the lattice system with the periodic boundary conditions, namely, the partition function of the torus-lattice system. The free-energy of the plane square-lattice system and the free-energy of the torus-lattice system are respectively given by

$$F = -k_B T \ln Q \tag{4}$$

$$F_1 = -k_B T \ln Q_1 \tag{5}$$

Because of the exponential function property, we have

$$Q\rangle 0, \quad Q_1\rangle 0, \quad Q\rangle Q_1 \tag{6}$$

the difference between F_1 and F is

$$\Delta F = F_1 - F = k_B T \ln(\frac{Q}{Q_1})\rangle 0 \tag{7}$$

Under the thermodynamic limit condition ΔF becomes infinitesimal but non-zero, which makes the critical temperature of the torus-lattice system infinitely close to the critical temperature of the plane square-lattice system so that Onsager's solution can be regarded as an exact solution of the plane square-lattice system. The infinitesimal difference $\Delta F\rangle 0$ shows that the original state cannot change into the state with the periodic boundary conditions in the thermodynamic equilibrium state, and the small offset of the free-energy makes the transformation of the topological structure of the system so great that its fundamental group is completely different from the original[3]. The thermodynamic limit guarantees the systems with and without the periodic boundary conditions have the same differential topology so as to have the same property of the continuous phase transition. This is not only seen in the plane-lattice system, but also will be seen in the one-dimensional lattice system [2]. Furthermore, if the spin states of all lattices at the boundary are the same [which is one kind of the periodic boundary conditions] these lattices themselves will gather up to one lattice to make the plane-lattice system become a sphere-lattice system. The solution of such a system will also be equivalent to Onsager's solution because of its periodic boundary conditions, and there exists certainly the continuous phase transition. Its geometric topology shows the sphere surface is simply connected[3], which allows the system to be ordered. However, there is no non-vanishing vector field on the sphere surface because of the "hairy ball theorem"[3], which is distinct from the torus-lattice system. When the sphere-lattice system becomes ordered its total spin-projection orientation will be along the sphere radius, which is arbitrary.

The second sense of the influence of the geometric properties is that the ordered system is not equivalent to its appearance of continuous phase transition all the time. A one-dimensional Ising model can be constructed on each site, with coordinate positions $x = 0, +1, -1, +2, -2, ...$, on which is laid one lattice spin with free degree $n = 1$. Such a system can be referred as to as an Ising model on the real line. Using the periodic boundary conditions, it is proved that the 1-dimensional system has no continuous phase transition. The periodic boundary conditions changed the real line into a circle embedded in a plane, which means that the circle-lattice system also cannot have the continuous phase transition like the real-line-lattice system because of the thermodynamic limit condition. However, the real-line–lattice system can become ordered because it is simply connected[3]. In other words, the whole lattice system can be looked at as a lattice with the total spin, but there is no non-vanishing vector field. The circle is not simply connected, and it allows the non-vanishing vector field to exist, which direction is clockwise or anticlockwise. Here, we noticed that the 1-dimensional lattice system with or without the periodic boundary conditions have the same differential topology but the geometric topology makes them appear in different spin-projection orientations. The key point to cause the system to become ordered is its geometric topology, not its differential topology. While the free degree of spin changes from $n = 1$ into $1\langle n \langle +\infty$ [the chain–lattice system can also become ordered without the continuous phase transition [2]]. We cannot say that all these phenomena are physical abnormalities for all possible values of n. As a matter of fact, they just show a common regular pattern; namely, agreement with their geometric and differential topology parameters.

Last, the spin-projection orientations also depend upon the dimensionality of the embedding space for the lattice system. If the embedded space is 2-dimensional, the spin-projection orientation is forbidden in the plane's normal direction for a plane lattice system or the circle-lattice system. The torus must be embedded in 3-dimensional space, and its non-vanishing vector field is also 2-dimensional at least. In the 1-dimensional space the spin-projection orientation normal to the real line cannot exist.

3. Conclusion

In summary, we found that the differential topology of the lattice-system of the Ising model determines whether there can be a continuous phase transition,The geometric topology of the space the lattice-system is embedded in determines whether the system can become ordered. Also if the system behaves ordered it may not have a continuous phase transition. Thus the spin-projection orientations are strongly influenced by the geometric topology of the lattice-system embeddeding space in.

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EJTP **7** (2005) 20–25 *Electronic Journal of Theoretical Physics*

Simulation of Ginger EPR Spectra Obtained by X-Irradiation: Quantum Approach

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Received 1 July 2005, Published 20 August 2005

Abstract: The ginger sample has been exposed to X-rays at cumulative doses. The foodstuffs irradiation is used in particular to improve their hygienic qualities and increase their shelf lives. This process has been approved by various international organizations: FAO – AIEA – WHO. In the present work, we propose to reproduce by simulation, based on a quantum approach, of the ESR (Electron Spin Resonance) spectra. The semi-classical approach is valid for a simple system, but not for a complex system such as an atom with hyperfine structure. In this case a quantum approach, based on spin Hamiltonian, is essential to interpret the ESR spectra. The main result is that the simulated spectra are in good agreement with the experimental ones obtained before and after irradiation.

Keywords: Simulation; ESR; Spin Hamiltonian; Ginger
PACS (2003): 76.50.1g; 02.30.Nw; 85.75.-d

1. Introduction

Electron paramagnetic resonance (EPR), also called electron spin resonance (ESR), was discovered in 1944 by Zavoisky [1] and has assumed an increasingly prominent position in various fields, especially in biochemistry-biology [2], chemistry [3], analysis of materials [4-6], geology and in medicine, or even in the two dimension imagery domain [7]. The phenomenon is based on the magnetic moment of an unpaired, spining free electron.

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In a magnetic field, the unpaired electron, which has a spin quantun number S =1/2 ($m_s = \pm 1/2$), precesses about the field axis (z-axis) with a component of its spin angular momentum either parallel (m_s = +1/2) or antiparallel (m_s = -1/2) to the z-axis. An oscillating magnetic field at right angles to the field axis induces transitions between the two spin states when the frequency of the field is at or near the Larmor frequency of the precessing electron. It is important to note that the EPR technic is an efficient and non-destructive method which is often applied to the quality control of irradiated foodstuffs [8-14]. Among species that are susceptible to give an EPR signal, we quote the transition-metal ions (oxidization state presenting the unpaired electrons), the organic radicals (CH_3,..), some inorganic radicals at the stable state (NO_2, NO, ClO_2, ..) and the unstable atoms (H, OH, ..). This method presents the advantage of important sensitivity which allows to detect 10^{11} unpaired electron spins by gram.

Recalling that, in a recent experimental work [15], a comparative analysis between ginger spectra observed before and after X-irradiation, permitted to follow the evolution, according to the dose absorbed by the sample, of the spectra shape and the amplitude. The signal of the unirradiated sample is characterized by a g-value of 2,0032 ± 0,0005, whereas after irradiation the observed signal present a weakly pick centered on the g = 2,0028 ± 0,0005. These results show that free radicals are produced by irradiation in ginger.

To complete this study, we are concerned with simulation, based on a quantum approach, the ESR spectra obtained before and after irradiation of ginger. We found, in particular, that simulated spectra are in good agreement with the experimental ones notably after irradiation.

This paper is organized as follows : In section 2 we present a theoretical model, we show also the spectra simulation and their comparison with those obtained by the experience [15]. We draw some concluding remarks in section 3.

2. EPR spectra simulation

The spin Hamiltonian of ginger sample, expressed in energy units, (assuming no hyperfine interaction) is given by :

$$H_{spin} = g\beta \mathbf{H}_0 \mathbf{S} \tag{1}$$

Where $\mathbf{H}_0$ is the external magnetic field vector, β denotes the Bohr magneton, g is the splitting factor and $\mathbf{S}$ represents the electron spin operator.

Using spin functions based on the quantum number m_s, equation (1) can be used to compute energy levels. Equating energy differences for the allowed transitions ($\Delta m_s = \pm 1$) with the microwave photon energy,

$$E(m_s + 1) - E(m_s) = h\nu \tag{2}$$

with

$$E(m_s) = g\beta H_0 m_s \tag{3}$$

For a given frequency of radiation, ν absorption occurs at the resonant magnetic field, H_r, in units of Gauss, given by

$$H_r = \frac{h\nu}{\beta g} \tag{4}$$

Here h is the Planck's constant.

The radiofrequency microwave field $\mathbf{H}_1$ must be perpendicular to the external field $\mathbf{H}_0$. For convenience, $\mathbf{H}_1$is taking along z-direction.

The transition probability between two states $|i\rangle$ and $|f\rangle$ can be written as :

$$P_{if} = \frac{|<f|\beta H_1 g S|i>|^2}{4\hbar^2} f(\nu - \nu_{if}) \tag{5}$$

with $f(\nu - \nu_{if})$ is the lineshape function, ν_{if} is the resonance frequency and H_1 is magnetic field amplitude of the radiation.

The lineshape function is expressed here in frequency. For experimental reasons, however, the microwave frequency is usually held constant in EPR experiment and the magnetic field is swept linearly. So, we must substitute the lineshape function $f(\nu - \nu_{if})$ by an another equivalent function F expressed in the terme of the magnetic field [20]

$$F(H - H_r) = \frac{2\Delta H}{\pi(\Delta H^2 + (H - H_r)^2)} \tag{6}$$

where $\Delta H$is the full width at half intensity and H_r represents the resonance field.

F is related to f by the relation [21]:

$$f(\nu - \nu_{if}) = |\frac{\partial H_r}{\partial \nu}| F(H - H_r) \tag{7}$$

Combining relations (5) and (7), the transition probability becomes :

$$P_{if} = \frac{|<f|\beta H_1 g S|i>|^2}{4\hbar^2} |\frac{\partial H_r}{\partial \nu}| F(H - H_r) \tag{8}$$

3. Results and Discussion

Recall that, the ESR spectra of unirradiated and X-irradiated sample were recorded at room temperature with the exposition period of 12 h 30 min [15]] by EMS 104 (BRUKER), (X-band) spectrometer [16].

In figures 1 and 2, we show the simulation results of ESR spectra compared by experimental ones. The shape of the measured spectra after irradiation is comparable to those relative to the paprika [17] or red pepper [18] samples γ-irradiated.

The classical analysis [19] using the macroscopic equations gives an excellent approach to magnetic resonance, particularly because it gives a simple physical picture. It is rather less applicable to electronic paramagnetic than to nuclear paramagnetic substances because the case where the only major interaction is that with the external applied field tends to be the exception rather than the rule.

This approach is valid for a simple system, such as an atom with an electronic but no nuclear magnetic moment, or vice versa

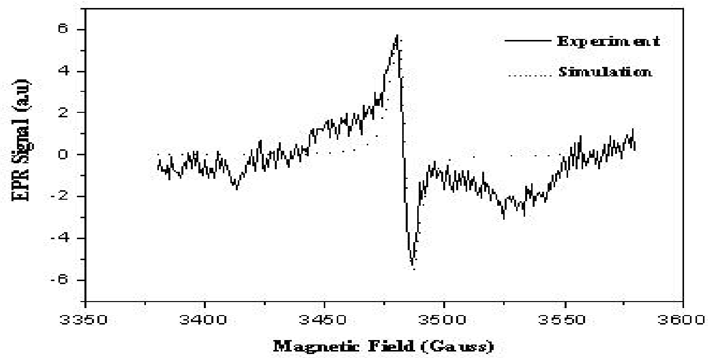

Fig. 1 ESR spectra before irradiation at room temperature

For a complex system such as an atom with hyperfine structure the semi-classical approach is not valid. In this circumstance a quantum mechanical approach is essential, and the resonance spectrum can only be interpreted in terms of spin Hamiltonian. The interaction considered in our simulation program for the ginger sample is the electronic Zeeman interaction which is the interaction of the magnetic moment of the electron with the externally applied magnetic $\mathbf{H}_0$.

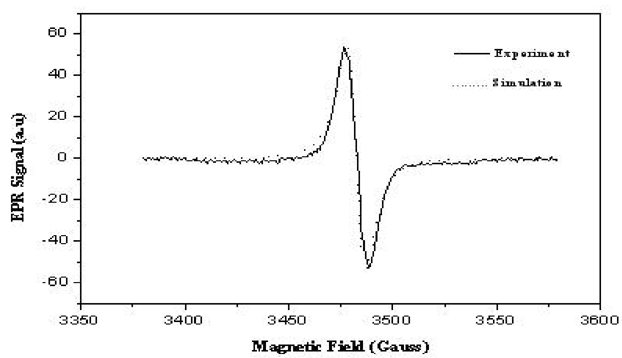

Fig. 2 ESR spectra 6 hours after irradiation at room temperature

We notice that the EPR spectra obtained by simulation are in good agreement with the experimental ones notably after irradiation. The parameters for the spectra simulation calculations shown in fig. 1 and 2 are given in Table 1.

Spectral parameters	*Before irradiation*	*After Irradiation*	*References*
$g - values$	2,0032	2,0028	[15]
$g - values$	2,0033	2,0029	*our parameters*
Resonance field (G)	3483,17	3482,69	*our parameters*

Table 1. Simulation parameters of ginger sample at room temperature.

3. Conclusion and Perspectives

In this work, we are concerned with simulation, based on quantum approach, of ESR spectra of ginger before and after X-irradiation. The ESR spectra are generated basically by finding the values of the magnetic field H_r at which, at a given value of the microwave frequency ν , there is a transition between energy levels. At that value, a line is produced by a lineshape function, typically Gaussian or Lorentzian. We determined the values of the spin Hamiltonian parameters for which the calculated spectrum best matches the observed spectrum. These parameters are not usually critical to the identification of the radicals that are present in irradiated foodstuffs.

The work concerned with simulation of the spin trap ESR (electron paramagnetic resonance) spectra is in progress. .

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EJTP **7** (2005) 27–46 *Electronic Journal of Theoretical Physics*

Quantum AdS^{1+3} Black Holes with Effective Cosmological Constant

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Received 3 Aug 2005, Published 20 Aug 2005

Abstract: A quantum **AdS**$^{1+3}$ massive and massless black holes with effective cosmological constant induced from non-minimal coupling and supergravities arguments are constructed and discussed in details.

Keywords: effective cosmological constant, non-minimal coupling, massive and massless black holes
PACS (2003): 04.62.+v, 04.70 Bw, 04.70 Dy.

1. Introduction

It was a surprising but interesting result when in 1992-93, Bañados et al. showed that 1+2 "black hole" with negative cosmological constant could be theoretically produced. Such Einstein's field equations solutions are in fact locally isometric to **AdS** constant curvature background [**1,2**]. In general, as it was noticed in [**3**], due to its peculiar asymptotic space with a timelike boundary at spatial infinity, the construction will be possible whatever the dimension of space-time is. Making **AdS** "black hole" in 1+3 dimensions is possible. In this way, S. Aminneborg et al. obtained isometric **AdS** black holes with event horizon that are tori or Riemann surfaces of genus higher than one, with one or two asymptotic regions [**3**]. In this paper, we will produce "*quantum massless-black hole*" from non-minimal coupling and supergravities arguments, with negative effective *quantum* cosmological constant. By "*quantum*" we mean the presence of the Planck constant "$\hbar$" in our field equations and by "*massless black hole*", we mean zero **AdS** space with

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zero Schwarzschild mass. But also some physical properties (*in particular temperature and entropy*) of massive (*free-charged and charged*) Schwarzschild-de-Sitter black holes (**SdS**) will be discussed through this paper. Although **AdS** spacetime (*generically arose as ground state in supergravity theory*) does not seem to correspond to the world in which we live, its importance has been noticed in many occasions. In fact, the presence of a negative comological constant makes it possible for a black hole to reach stable thermal equilibrium with a heat bath [**4**].

In fact the idea of massless black holes is not new. They are considered as objects with strange properties. In particular, they are supersymmetric, they saturate all "bounds" and "anti-bounds", they are annihilated by all supersymmetric and Lorentz charges and they are of Minkowski type being a vacuum state [**5,6,7,8,9,10**]. In General Relativity (**GR**), when the mass of the Schwarzschild metric goes to zero, the space-time approaches Minkowski's. However, things are different for the Reissner-Nordström background. It is well-known that in this case, the metric obtained describes a charged massless system with naked singularity at the origin. This singularity could be remedied if for example, we change the sign of the charge was Wick-rotated, described a genuine black hole solutions of the Einstein-anti-Maxwell theory [**11**]. In addition, despite the successes of Hawking quantum black hole theory, we are left with the feeling that the quantum story is not complete [**12,13,14,15,16**].

There exist today a lot of interest concerning density perturbations and black hole formation in hybrid two-stage inflations [**17,18,19**]. It was proved that quantum fluctuations at the time corresponding to the phase transition between the two inflationary stages leads to the formation of a large number of inflating topological defects. It was also showed that density perturbations in hybrid inflation models of the new type can be very large on the scale corresponding to the phase transition [**20**]. The resulting density inhomogeneities lead to a production of black holes where numbers can be made extremely small, but in general it could be sufficiently large to have cosmological and astrophysical important implications (*for example, the dark matter in the universe*). It is also possible to have two-stage hybrid inflation without suppressed the production of black hole, but where *their typical masses are very small, even massless*. Such models lead to a completely different thermal history of the universe, where post-inflationary reheating occurs via black hole evaporation [**20**]. The existence of such small-mass or massless black holes represents our first motivation in this work.

Another motivation came from the deepest and important lessons that we have learned over the past modern quantum decade is *that there is no fundamental difference between elementary particles and black holes*. As mentioned a lot of time by 't Hoof, "*black holes are the natural extension of the elementary particle spectrum*". This is especially clear in string theory where black holes are simply highly excited string states[2] [**21,22,23,24**]. Remember that the quantum consistency of string theory requires that extremal black

[2] Zero mass extreme black holes play also a remarkable role in the non-perturbative aspects of string theory; their role in order to understand the conifold singularities that appear in the low energy Lagrangian describing the moduli space of Calabi-Yau vacua of (*type II*) string theory is crucial.

holes be treated as elementary quanta.

An additional motivation came from recent works relating the cosmological constant to the black hole physics. It was known that there is a quantum mechanical tunneling process which, through nucleation of a membrane, induces a transition between two **dS** spaces, lowering the cosmological constant up to zero leaving a black hole behind [**25,26,27,28**] (*in general, a negative cosmological constant allows the existence of black holes with a topology* $\Re^2 \times \Sigma$*, where* Σ *is a two-dimensional manifold of constant curvature* [**29,30,31**]).

Finally, remember that in black hole topology, the quasinormal modes, which are typically characterized by a spectrum that is independent of the initial conditions, are a sort of fingerprint of the black hole depending only on its parameters and on the fundamental constants of the system. There exists a lot of interest on the study of these quasinormal modes in asymptotically flat spacetimes with negative cosmological constants, in particular massless black hole with massive scalar field non-minimally coupled to the curvature, with the horizon geometry assumed to have a negative constant curvature [**28,32,33,34**].

Through the **AdS/CFT** (***C****onformal* ***F****ield* ***T****heory*) correspondence the quasinormal modes can be related to the relaxation time scale of the associated thermal states[3] [**35,36**]. Recently, a connection has also been conjectured between the quasinormal modes and critical phenomena of black hole formation in an asymptotically **AdS** background [**37,38**]. Detailed analysis showed that the energy-momentum flux density of these models vanishes at the asymptotic region when the square of its effective mass is less than zero.

In this work, we explain a new approach describing in particular *static massless quantum black hole* (**SmQBH**) and show that these exotic objects could have important physical features.

2. Modified Scalar Curvature From Non-Minimal Coupling

In **GR**, the cosmological constant describes the energy density of the vacuum (*empty space*), and it is a potentially important contributor to the dynamical history of the Universe. In fact, it is not well theoretically understood, what is the physical mechanism which cancels the vacuum energy through the different phase transitions that our universe undergoes! On the other hand, recent observations of indicates that the Universe is in accelerated regime [**39,40,41**]. If theoretically, the total energy of the universe consists only of ordinary matter and dark matter, than one can interpret the dark energy as the vacuum energy corresponding to the cosmological constant, that is Λ or as the slowly changing energy of a certain scalar field with a vacuum ϕ corresponding to the equation

[3] The interest on **AdS** spacetime was revived by a conjectured duality between string theory in the bulk of **AdS** and conformally **CFT** living on the boundary of **AdS**. The **AdS/CFT** correspondence gives an explicit relation between Yang-Mills theory and string theory. More recently, there has been a renewed interest in **AdS** spacetime since progresses in theories of extra dimensions present us with the enticing possibility to explain some long-standing particle physics problem by geometrical means.

of state $p = -\rho$ [**42**]. In both cases the Universe is accelerated with time and approaching de-Sitter (**dS**) regime. From the point of view of string theory, any dimensional parameter must be expressed in terms of the fundamental string scale, and of vacuum expectation values of scalar fields, so that the physics of the cosmological constant is nothing than the physics of the corresponding scalar fields. This complicates the situation, because it is a difficult task to envisage string theory in the context of a non exotic cosmological constant. The corresponding spacetime is the **dS** one having as we know an event horizon (*see S. Weinberg proof of no-go theorem in* [**42**]).

In fact, the most natural way to talk about a vanishing cosmological constant came from symmetry argument. Supersymmetry (**SUSY**) predicts that the masses of boson and fermions are equal, and this must be broken. But in the context of cosmology, we need the local version, which is supergravity (**SG**). Although **SUSY** may be broken while the cosmological constant remains too small, we lost our rational for a vanishing cosmological constant and undesired fine-tuning reappears again.

In most of the theoretical models of dark energy it is assumed that the cosmological constant is equal to zero and the potential energy of the scalar field driving the present stage of acceleration, slowly decreases and eventually vanishes as the field rolls to$\phi = \infty$ [**43,44,45,46,47**]. As a result, after a transient **dS**-like stage, the speed of expansion of the Universe decreases, and the Universe reaches Minkowski regime. Of course, depending on the choice of the model, the flat Universe will become **dS** space, or Minkowski space, or collapse [**48,49,50,51**]. However, it was found that one can describe dark energy in some extended supergravities that have a **dS** solutions [**52,53**]. These **dS** solutions correspond in general to the extrema of the effective potentials $V(\phi)$ for some scalar fields ϕ . An interesting result of these solutions is that the squared mass of these scalars in all theories with $N = 2$ (*extended supergravities with unstable **dS** vacua*) is quantized in units of the Hubble constant H_o . That is $m^2 = nH_o^2$ where n are integers of order of unity (*in units of unity Planck Mass*). In extended supergravities with a positive cosmological constant, one always has $3m^2 = n\Lambda$ where Λ being the cosmological constant. For the $N = 8$ supergravity, **dS** vacuum corresponds to an unstable maximum $m^2 = -6H_o^2$ at $|\phi| << 1$ and $V(\phi) = 3H_0^2(1 - \phi^2)$. Meanwhile for $N = 2$ gauged supergravity with stable **dS** vacuum, one has $m^2 = 6H_0^2$ for one of the scalars and in this case $V(\phi) = 3H_0^2(1 - \phi^2)$ [**54,55,56,57**]. Note here that m is not in these theories, a massive cosmological graviton. In [**58,59**], it was showed that the graviton has a mass in **AdS** space-time given by $m_g^2 = -2\Lambda/3$. The authors were able to show that a perturbation of the Einstein equations, with the presence of the cosmological constant, in a **dS** or **AdS** background, produces exactly the equations obtained from the Fliers formulation for the massive spin-2 field. Their final conclusion is that if the graviton has a real mass not null, than the cosmological constant should be negative.

In application to the cosmological constant problem, this leads to the conclusion that there are ultra light scalars with the mass of the order $m \simeq H \simeq 10^{-33}eV$. The significance of this fact and the possibility to use these supergravity models in modern cosmology still have to be well studied and understood. The existence of such ultra light fields may be a desirable feature for the description of the accelerated universe. Their presence signals that the corresponding potentials are very shallow. In extended supergravity theories ultra light fields necessarily come in a package with too small Λ. Due to the presence of Λ , **SUSY** in **dS** vacua is broken spontaneously, the scale of **SUSY** breaking here is $10^{-3}eV$. Before it is coupled to a '*visible sector*', both the tiny Λ as well as the ultra light masses of scalars, that is m , are protected from large quantum corrections.

Coupling of these theories to real universe is a big problem, of course. If they play a role of a '*hidden sector*', one may ask whether the tiny $m \simeq H$ will be preserved after coupling to the '*visible sector*'. The preservation of the small Λ may imply preservation of small scalar masses m. Thus, extended supergravities suggest a new perspective for investigation of the cosmological constant problem, intertwined with ultra-light scalars [**60,61**].

Although the non minimal coupling significantly modifies quantum geometry, the highly non-trivial consistency checks for the emergence of a coherent description of the quantum black hole horizon continue to be met.

In fact, the generalization of the Klein-Gordon equation in curved spacetime includes the possibility of an explicit non-minimal coupling between the scalar field driving inflation and the Ricci curvature of spacetime. Nowadays, there are many reasons to believe that a non-minimal coupling term is present in the Klein-Gordon equation. Quantum corrections generate a non-minimal coupling even if it is absent in the calculation. A nonzero value for this later is also required in order to normalize the theory. Furthermore, it has been argued that a non-minimal coupling term is to be expected whenever the spacetime curvature is large. Due to these considerations, it seems sensible but important to consider an explicit non-minimal coupling between the scalar field ϕ and the scalar curvature R in the inflationary paradigm.

In a recent paper [**61**], we introduced, for some scalar field ϕ , a non-minimal coupling between the scalar curvature and the density of the scalar field in the following form $L = -\xi\sqrt{g}R\phi^*\phi, \xi = 1/6$. R is the scalar curvature and ϕ^* is the complex conjugate of ϕ. From a view point of quantum field theory in curved space-time, it is natural to consider such a non-minimal coupling. In fact, the case where $\xi = 1/6$ results in an extension of the property of conformal invariance for massless fields, which is attractive from physical point of view. This parameter describes the strength of the coupling between the curvature of spacetime and the inflaton. Minimal coupling corresponds to $\xi = 0$. It was shown that in this case and for a particular scalar negative complex potential field $V(\phi\phi^*) = 3/4m^2(\omega\phi^2\phi^{*2}-1)$,$\omega$ being a tiny parameter [**62-71**], inspired from supergravity inflation theories, ultra-light masses "m" are implemented naturally in Einstein field equations (**EFE**), leading to a cosmological constant "Λ" in accord with observations[4]. In matter-

[4] It has been argued that a non-minimal coupling term-generated by quantum corrections-is to be expected whenever the space-time curvature is large; in most theories that describe inflationary scenarios, it turns out that a value of ξ different from zero is unavoidable. As a matter of fact, it seems sensible to consider an explicit non-minimal coupling in the supergravities inflationary paradigm.

free background, the scalar curvature was found to be $R = 4\bar{\Lambda} - 3m^2 = 4\bar{\Lambda}$ where $\bar{\Lambda} = \Lambda - 3/4m^2$ is the *effective cosmological constant*[5] (*in natural units, where " " is the Planck constant and " " being the celerity of light*). In this particular case and for $\omega << 1$, a possible candidate field equations for the scalar curvature R , will be:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \left(\Lambda - \frac{3}{4} m^2 \right) g_{\mu\nu} = 0 \tag{1}$$

We define $\bar{\Lambda} = \Lambda - 3m^2/4 \equiv \Lambda_1 + \Lambda_2$ or in natural unit $\bar{\Lambda} = \Lambda_1 - 3m^2c^2/4\hbar^2$ as the effective cosmological constant (*one can says that we are dealing here with a theory with two cosmological constant*). Here Λ_1 (*the first*) is the standard, that is Einstein's cosmological constant and Λ_2 (*the second, which is proportional to the inverse square of the Compton wavelength*) is our quantum (*so-called due to the presence of "$\hbar$"*) second negative cosmological constant. Note that if $\Lambda_1 = 0$, the scalar curvature is negative and the spacetime is not Minkowskian as in the standard model. When $\Lambda_1 = \Lambda_2, R = 0$. In this case, m is approximately of the same order of the graviton mass described in [**58,59**] (*which is an interesting fact*). But the difference is that this our quantum mass is viewed as a boson of spin zero, while the graviton is of spin 2. Remember that the conditions $\Lambda_1 + \Lambda_2 > 0$ or < 0 play important role in modern cosmological theories, in particular inflation [**63-70**], in particular when the mass m is of the order of the Hubble constant.

In what follows, we will be also interested on the case where one of the two cosmological constants, that is, the Einstein cosmological constant, is set equal to zero ($\Lambda_1 = 0$). The resulting equation combines gravity (*geometry*) with quantum theory ($\hbar$) (*another interesting fact*). With the presence of the cosmological constant, this equation and in particular our scalar curvature is similar (*only in form*) to the one obtained by J. Charon long time ago in his complex relativity [**72**], but of course his vision and motivations were totally different from ours. It is interesting to investigate how much the presence of the quantum ultra light masses (*in form of a quantum cosmological constant*) our quantum field equations will contribute and change in most of the static standard model, in particular the **AdS** one.

3. AdS^{1+3} Spacetime with Negative Effective Quantum Cosmological Constant and Zero Einstein's Lambda

The requirement of a static field implies as it is well-known that the components of $g_{\mu\nu}$ of the metric tensor shall not depend upon the time coordinate. The requirement of spherical symmetry can be expressed by writing the metric to be determined in the well-known form [**73,74**]:

$$ds^2 = g_{00}(r) c^2 dt^2 + g_{11}(r) dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \tag{2}$$

We work with metric signature $(+ - --)$ and we choose naturally $g_{11}g_{00} = -1$, with $g_{00} = 1 + F$ where F is a spherically symmetric time-independent perturbation. The Ricci tensor and Riemann scalar are found in any standard **GR** textbook:

[5] In [**61**], $8\pi G \equiv \kappa$ was set equal to unity.

$$R_{00} = -\frac{c^2}{2}(1+F)\left[F'' + 2\frac{F'}{F}\right] \tag{3}$$

$$R_{11} = \frac{1}{2(1+F)}\left[F'' + 2\frac{F'}{F}\right] \tag{4}$$

$$R_{22} = F + rF' \tag{5}$$

$$R_{33} = sin^2\theta\left[rF' + F\right] \tag{6}$$

$$R \equiv g^{\mu\nu}\ R_{\mu\nu} = -\left[F'' + 4\frac{F'}{r} + 2\frac{F}{r^2}\right] \tag{7}$$

$$R_{\mu\nu} \neq 0, \mu \neq \nu \tag{8}$$

By choosing $F = ar^2$, where a is a constant, we find using equation (7), $R = -12a \equiv -3m^2c^2/\hbar^2$, so that$a = m^2c^2/4\hbar^2$. The metric the takes the following form:

$$ds^2 = c^2\left(1 - \frac{\Lambda_2 r^2}{3}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{\Lambda_2 r^2}{3}\right)} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

$$= c^2\left(1 + \frac{m^2c^2}{4\hbar^2}r^2\right)dt^2 - \frac{dr^2}{\left(1 + \frac{m^2c^2}{4\hbar^2}r^2\right)} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{9}$$

describing an quantum **AdS** with negative effective quantum cosmological constant. Note that in terms of Compton wavelength, the metric (9) can be put in the following form:

$$ds^2 = c^2\left(1 + \frac{r^2}{\tilde{\lambda}_C^2}\right)dt^2 - \frac{dr^2}{\left(1 + \frac{r^2}{\tilde{\lambda}_C^2}\right)} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{10}$$

where $\tilde{\lambda}_C = 2\lambda_C$ represents the **AdS** radius. Note that if $\Lambda_1 \neq 0$, $g_{00}(r) = 1 - 1/3\left(\Lambda_1 + \Lambda_2\right)r^2$, so that if $\Lambda_1 > 3m^2c^2/4\hbar^2$, the metric is of **dS** type, while in case $\Lambda_1 < 3m^2c^2/4\hbar^2$, it is of **AdS** type.

For *massive* black hole (*Schwarzschild-de-Sitter spacetime* (**SdS**)), the potential $g_{00}(r) = 1 - 2MG/rc^2 - 1/3(\Lambda_1 - 3m^2c^2/4\hbar^2)r^2$ where M is the Schwarzschild. There are two positive roots of the cubic equation $g_{00}(r) = 0$ corresponding to black horizon r_+ and cosmological horizon r_c. The third root locates at $r_- = -(r_+ + r_c)$. For $\Lambda_1 > 3m^2c^2/4\hbar^2$, as the Schwarzschild mass increases, the black hole radius r_+ increases and the cosmological horizon r_c decreases monotonically. For maximum value of M, $r_+ = r_c = \sqrt{1/\Lambda_1 + \Lambda_2}$, which is the biggest black hole that can be formed in de Sitter space. Using the results for highly damped quasi-normal modes, one can obtained the area and entropy spectrum of event horizon [**75,76,77,78**]. Near external **SAdS** black hole, provided $\Lambda_1 > 3m^2c^2/4\hbar^2$, the quantum area is found to be of the form $\Delta A = 24\pi\hbar\sqrt{v_0/\kappa_{BH}^2 - 1/4}$ where $v_0 = \kappa_{BH}^2 l\,(l+1)$, $\kappa_{BH} \approx (r_c - r_+)/2r_+^2$ is the surface gravity, l being the angular momentum [**79**]. This quantum area seams to be not universal for all black holes [**80,81,82**]. In case $\Lambda_1 < 3m^2c^2/4\hbar^2$, we have *Schwarzschild-Anti-de Sitter spacetime* (**SAdS**). For large **SAdS** black hole ($r_0 << r_+$ *where* $r_0 = \sqrt{3/\Lambda_1 + \Lambda_2}$), we find from

$g_{00}(r) = 0$ that the relation between the cosmological constants $\Lambda_1$and Λ_2 and black hole horizon r_+ is $\Lambda_1 - 3m^2c^2/4\hbar^2 = -6MG/r_+^2c^2 + 3/r_+^2$. For a very small black hole, and by considering a system consisting of initially well-separated matter and black hole, the first law of thermodynamics gives approximately the temperature $T = 1/4\pi\zeta r_c$, ζ is constant (*providing that* $\Lambda_1 > 3m^2c^2/4\hbar^2$). The entropy is given by $S = \pi r_+^2 = 1/4A$ where A is the area of the event horizon. This is the same form of entropy s in asymptotically flat space. Phase transitions of thermal radiation in **SAdS** will be dealt in future work.

Concerning the extreme limit for massive black hole, it has been investigated by several authors [**83,84,85,86,87**]. An interesting idea is to introduce a non-extremal parameter ε than perform the following coordinate change $r = r_{ext} + \varepsilon r_1$, $t = \varepsilon t_1$ with $\varepsilon << 1$ in the near-extremal limit. One then can show that a large class of black hole solutions admits a well defined limit procedure, in partic731 the massive **SdS** and **SAdS** spacetime. Following [**85**], we prove that in the **SdS** case, the final solution is locally $dS_2 \times S_2$ and is equivalent to the Nariai solution [**88**], but it's a cosmological solution with $\Lambda_1 > 3m^2c^2/4\hbar^2$, while for **SAdS** where $\Lambda_1 < 3m^2c^2/4\hbar^2$, the limiting final metric will be $AdS_2 \times \Sigma_2$, Σ_2 being a Riemann surface.

Turning now to the *massless* case, the quantum metric (9) has no singularities and its geometry has constant negative curvature$R = -3\lambda_C^{-2}$ (*assuming the mass m is constant and* $m^2 > 0$). There is no intrinsic horizon of this space-time, from which it follows that the surface gravity yields a divergence. The metric is regular and globally homeomorphic to $AdS_2 \times S_2$, or geometrically, this solution is viewed as the direct product of two 2-dimensional manifolds [**89,90**]. Following [**91,92**], making massless quantum or microscopic **AdS** black hole seems possible. In fact, formulating quantum entropy of black hole from non-minimal coupling is not new [**93,94**]. But our approach here is different and indirect. We also note that according to Rindler (*see* [**59**]) and Deser et al. [**58**], detectors with constant acceleration "$\hat{a}$" in **dS** and **AdS** spaces with cosmological constant Λ measure temperatures $2\pi T = (\Lambda/3 + \hat{a})^{1/2} \equiv \hat{a_5}$, this later being the *5-acceleration* in the embedding flat 5-space.

For **dS** background, this works correctly, while for **AdS** background, the physics is quite different leading to measure imaginary temperature (*the temperature is well defined down to the critical value* $\hat{a_5} = 0$ *, again in accord with the underlying kinematics*).

The resolution of this paradox is that its motion becomes spacelike. As mentioned in [**95**]*, these considerations were confirmed by explicit calculations of the correlators of a quantum field in* ***AdS****, whose result was independent of the different boundary conditions permitted by the* ***AdS*** *causality ambiguities, and led to the standard Unruh Planckian temperature distributio*n, putting our quantum massless **AdS** in safe.

4. Quantum Massless AdS Black Hole and the 3-dimensional Quantum Relativistic Oscillator

In [**96**], it was shown that a static (1+3) **AdS** metric defines naturally a relativistic harmonic oscillator (**RHO**) in Minkowski space. We follow their arguments and we set the lagrangian of our metric (18) in the following form:

$$L = -mc\sqrt{1 + \frac{r^2}{\tilde{\lambda}_C^2} - \frac{V^2}{c^2} + \frac{1}{c^2\tilde{\lambda}_C^2}\frac{(\vec{x}\cdot\vec{V})^2}{1+\frac{r^2}{\tilde{\lambda}_C^2}}} \tag{11}$$

where m is the reduced mass of the system and V being the velocity. The geodesics can be derived from this equation. This Lagrangian defines a 3-dimentional quantum relativistic oscillator in Minkowski space. The Hamiltonian is given by:

$$H^2 = \left(1 + \frac{1}{\tilde{\lambda}_C^2}\right)\left(m^2c^4 + p^2c^2 + \frac{c^2}{\tilde{\lambda}_C^2}(\vec{x}\cdot\vec{p})^2\right) \tag{12}$$

$\vec{p}$ being the momentum vector. In fact, the Schrödinger equation associated with this Hamiltonian can be transformed, with a particular normal-ordering prescription, into a Klein-Gordon equation.

In our model, the Klein-Gordon (ϕ) and Schrödinger wave functions (ψ) are related by $\phi = \sqrt{1+\frac{r^2}{\tilde{\lambda}_C^2}\psi}$. The Klein-Gordon equation in our quantum **AdS** background is:

$$\left(\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}(1-3\xi)\right)\phi = \left(\partial_\mu\partial^\mu + \frac{m_{eff}^2c^2}{\hbar^2}\right)\phi = 0 \tag{13}$$

with $m_{eff}^2 = m^2(1-3\xi)$, where we have used the fact that $R = -3m^2c^2/\hbar^2$. Here $\partial_\mu\partial^\mu \equiv \Box$ stands for the Laplace-Beltrami operator and ξ is a numerical factor. Following [**38**], one can prove that the energy-momentum flux density vanishes at the asymptotic region if $m_{eff}^2 > 9/4m^2$. The solution of equation (13) is done in [**38**]. It was shown that for an **AdS** background ϕ must be restricted to be a purely ingoing wave at the horizon, a reason why the energy-momentum flux density should vanish at the asymptotic region[6].

In our case, the Laplace-Beltrami operator is given by:

$$\Box = \frac{1}{\left(1+\frac{r^2}{\tilde{\lambda}_C^2}\right)}\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - 2\frac{r}{\tilde{\lambda}_C^2}\frac{\partial}{\partial r} - \left(1+\frac{r^2}{\tilde{\lambda}_C^2}\right)\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\tilde{L}^2}{r^2} \tag{14}$$

where

$$\tilde{L}^2 = -\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \tag{15}$$

is the orbital angular momentum operator. After solving the Klein-Gordon equation, one finds the following wave functions (*for the detailed of the calculations, the reader is referred to* [**96**]):

$$\phi(\vec{x},t) = e^{-i\tilde{c}t}Y_m^l(\theta,\varphi)\left(1+\frac{r^2}{\tilde{\lambda}_C^2}\right)^{-p/2} r^l\phi_l^p(r) \tag{16}$$

[6] Topological **AdS** black hole with negative constant curvature exist also for spacetime dimensions greater than 4, that is $d > 4$ and also for gravity theories containing higher order of the curvature (*see* [**38**] *and references therein*).

where $\tilde{c} = \tilde{\lambda}_C \omega$, ω being the angular velocity and p is an arbitrary positive parameter. The functions $\phi_l^p(r)$ verify the following hypergeometric differential equation:

$$\begin{aligned} &\Big\{(1-\rho)\rho\tfrac{d^2}{d\rho^2} + \Big(l + \tfrac{3}{2} - \big(l + \tfrac{5}{2} - p\big)\,\rho\Big)\,\tfrac{d}{d\rho} \\ &\qquad - \tfrac{1}{4}(p(p-2l-3) + l(l+3) - N^2 + 12\xi)\Big\}\,\phi_l^p(\rho) = 0 \end{aligned} \tag{17}$$

where $\rho = -r^2/\tilde{\lambda}_C^2$ and $N = mc\tilde{\lambda}_C/\hbar$. With the regularity condition at the origin and the square integrability of the wave functions, the energy spectrum is then given by:

$$\begin{aligned} E &= \left(\frac{3}{2} + 2n + l + \frac{1}{2}\sqrt{9 + 4\left(\frac{m^2c^2}{\hbar^2}\right)\tilde{\lambda}_C^2 - 48\xi}\right)\hbar\omega \\ &= \left(\frac{3}{2} + 2n + l + \frac{1}{2}\sqrt{9 + 4\left(\frac{\tilde{\lambda}_C^2}{\lambda_C^2}\right)\tilde{\lambda}_C^2 - 48\xi}\right)\hbar\omega \\ &= \left(\frac{3}{2} + 2n + l + \frac{1}{2}\sqrt{25 - 48\xi}\right)\hbar\omega \\ &= \left(\frac{3}{2} + 2n + l\right)\hbar\omega + \left(\frac{1}{2}\sqrt{25 - 48\xi}\right)\hbar\omega \end{aligned} \tag{18}$$

where we have used the fact that $\tilde{\lambda}_C = 2\lambda_C$. For the ground state energy, we have:

$$E = \frac{3}{2}\hbar\omega + \left(\frac{1}{2}\sqrt{25 - 48\xi}\right)\hbar\omega \tag{19}$$

This quantum **AdS** black hole behaves as a relativistic harmonic oscillator. The system is exactly solvable.

Observe that when $\xi = 25/48$, the second term of equation (19) disappears, leaving the ordinary energy spectrum of 3-dimenstional non-relativistic oscillator, while in [**96**], this is obtained only in the non-relativistic limit, that is when . For this value of , the Klein-Gordon became:

$$\left(\partial_\mu \partial^\mu + \frac{\tilde{m}^2 c^2}{\hbar^2}\right)\phi = 0 \tag{20}$$

where $\tilde{m}^2 = -27/48m^2 < 0$ (*superluminal or sub-eV particles: tachyons*). For this value of ξ, the quantum theory is not well defined, if not at this level, not well understood. While for $\xi = 1/3$, $\Box\phi = 0$ (*massless Klein-Gordon scalar wave equation*) and $E = 3\hbar\omega$ for its corresponding ground states with angular velocity three times the standard one, that is $\omega^{'} = 3\omega$.

In conclusion, the energy spectrum coincides, up to the ground state energy, with that of the non-relativistic oscillator, but the presence of the mixing parameter could have, as we have seen, important impacts on the whole problem.

5. Charged Quantum AdS Black Holes, Tachyons and the Validity of the Quantum Theory

Due to its huge interesting physical aspects, much work has gone into deep understanding of charged **AdS** black holes (*thermodynamics, holography, quantum fluctuations...*), in particular, its quantum gravity features[7] [**12,97,98,99,100,101**]. This motivates us to study the impact of our theory on this later. When an electromagnetic fields is added to our field equations (1) as:

$$(T^{\mu\nu})_{e.m.} = \varepsilon_0 \left(F^{\mu}_{\alpha} F^{\alpha\nu} + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right) \tag{21}$$

where $F^{\alpha\beta}$ is the tensor of the electromagnetic field and ε_0 is the permittivity constant, we may found a metric in the centrosymmetric form of equation (2) with:

$$g_{00}(r) = 1 - \frac{2MG}{rc^2} + \frac{q^2 G}{4\pi\varepsilon_0 c^4 r^2} - \frac{1}{3}\left(\Lambda_1 - \frac{3m^2c^2}{4\hbar^2}\right) r^2 \tag{22}$$

$$\equiv 1 - \frac{2\tilde{M}}{r} + \frac{\tilde{Q}^2}{r^2} - \frac{1}{3}\left(\Lambda_1 - \frac{3}{4}\tilde{m}^2\right) r^2 \tag{23}$$

$$\equiv 1 - \frac{r_+}{r} - \frac{1}{3}\frac{r_+^3}{r}\left(\Lambda_1 - \frac{3}{4}\tilde{m}^2\right) + \frac{\tilde{Q}^2}{r^2} - \frac{\tilde{Q}^2}{r_+ r} - \frac{1}{3}\left(\Lambda_1 - \frac{3}{4}\tilde{m}^2\right) r^2 \tag{24}$$

where for simplicity, we set $\tilde{M} = MG/c^2$, $\tilde{Q}^2 = q^2G/4\pi\varepsilon_0 c^4$and $\tilde{m} = mc/\hbar$. is the Schwarzschild gravitational mass and G is the gravitational constant. When $\Lambda_1 < 3m^2c^2/4\hbar^2$, the metric corresponds to the Reissner-Nordström black hole with a negative effective cosmological constant. The mass of the corresponding black hole is given by:

$$M = \frac{1}{2}\left(r_+ - \frac{1}{3}\left(\Lambda_1 - \frac{3}{4}\tilde{m}^2\right) r_+^3 + \frac{\tilde{Q}^2}{r_+} \right) \tag{25}$$

and satisfies the laws of black hole thermodynamics with a temperature:

$$T_{BH} = \frac{1 - \frac{\tilde{Q}^2}{r_+^2} - \left(\Lambda_1 - \frac{3}{4}\tilde{m}^2\right) r_+^2}{4\pi r_+} \tag{26}$$

and a potential $\phi = Q/r_+$. In general r_+ and Q are independent, but in the extremal case, they get related as:

$$1 - \frac{\tilde{Q}^2}{r_+^2} - \left(\Lambda_1 - \frac{3}{4}\tilde{m}^2\right) r_+^2 = 0 \tag{27}$$

[7] The whole focus of the physics of charged black hole is to find physical mechanisms or a suitable quantum mechanics that resolve the Hawking paradox. It has seemed plausible to many black hole physicists that extremal charged black holes (**ECBH**) are the endpoint of Hawking evaporation if the black hole manages to retain its charge, that is, stabilize its charge. This particular and strange situation was ignored by most high-energy researchers. Because the formation and evaporation of **ECBH** can be studied within the framework of a controlled approximation scheme, we hope in the future to extract from them important information or novel physics in the aim to better understand quantum neutral black holes.

Note that when $\Lambda_1 = 3/4\tilde{m}^2$ and $r_+ = \tilde{Q}$, $T_{BH} = 0$. When dealing with cosmological Einstein-Maxwell theory, equation (22) takes the form:

$$g_{00}(r) = 1 - \frac{2\tilde{M}}{r} + \frac{\tilde{Z}^2}{r^2} - \frac{1}{3}\left(\Lambda_1 - \frac{3}{4}\tilde{m}^2\right) r^2 \tag{28}$$

where $\tilde{Z}^2 = \tilde{Q}^2 + \tilde{H}^2$ being the magnetic charge [**102,103**]. From the behavior of the curvature invariants:

$$R^2 = 16\left(\Lambda_1 + \Lambda_2\right)^2 \tag{29}$$

$$R_{mn}R^{mn} = 4\left(\frac{\tilde{Z}^4}{r^8} + \left(\Lambda_1 + \Lambda_2\right)^2\right) \tag{30}$$

$$C_{mnpq}C^{mnpq} = \frac{48}{r^4}\left(\frac{\tilde{M}}{r} - \frac{\tilde{Z}^2}{r^2}\right)^2 \tag{31}$$

one can remarks that such solutions possesses a single physical singularity located at $r = 0$. Here indices $m, n, ...$are "curved" world indices [**104**]. Note that when $\Lambda_1 = -\Lambda_2 = 3/4\tilde{m}^2$, $R = 0$. Also if the classical Einstein's cosmological constant is set equal to zero, keeping the other one, the curvature $R \neq 0$. In asymptotically flat space-time, that is $\Lambda_1 = -\Lambda_2 = 3/4\tilde{m}^2$, for $\tilde{Z}^2 < \tilde{M}^2$, there exists two horizons at radii $\tilde{\rho}_\pm = \tilde{M} \pm \sqrt{\tilde{M}^2 - \tilde{Z}^2}$. While for $\Lambda_1 \neq -\Lambda_2$, the condition $g_{00}(r) = 0$ is a quartic algebraic equation for r, that is:

$$-\frac{1}{3}\left(\Lambda_1 + \Lambda_2\right) r^4 + r^2 - 2\tilde{M}r + \tilde{Z}^2 = 0 \tag{32}$$

One may follows [**104**] and shows that for $\Lambda_1 > 3/4\tilde{m}^2$, the existence of an additional event horizon$\approx \sqrt{3/\Lambda_1 + \Lambda_2}$ constituting the "*outer edge*" of the **dS** universe in the given coordinate system. The Hawking effective temperature for the Reisner-Nordström solutions is given by:

$$T^{effective} = \frac{\kappa}{2\pi} = \frac{1}{4\pi\tilde{\rho}}\left|1 - \frac{\tilde{Z}^2}{\tilde{\rho}^2} - \left(\Lambda_1 - \frac{3}{4}\tilde{m}^2\right)\tilde{\rho}2\right| \tag{33}$$

where κ is the surface gravity at the horizon [**12**]. Taking $\tilde{Z}^2 = \Lambda_1 = 0$ gives the temperature of an uncharged cosmological **AdS** black hole. It is evident from equation (33) that horizons with zero Hawking temperature are located at simultaneous roots $\tilde{\rho}$ of both g_{00} and g'_{00} (*at double roots of* $g_{00}(\tilde{\rho})$). When such a double roots exists, $g_{00}(r)$ takes the following form:

$$g_{00}(r) = \left(1 - \frac{\tilde{\rho}}{r}\right)\left(1 - \frac{1}{3}\left(\Lambda_1 - \frac{3}{4}\tilde{m}^2\right)\left(r^2 + 2\tilde{\rho}r + 3\tilde{\rho}^2\right)\right) \tag{34}$$

with the corresponding critical relationships

$$\tilde{M} = \tilde{\rho}\left(1 - \frac{2}{3}\left(\Lambda_1 - \frac{3}{4}\tilde{m}^2\right)\tilde{\rho}^2\right) \tag{35}$$

$$\tilde{Z}^2 = \tilde{\rho}^2 \left(1 - \left(\Lambda_1 - \frac{3}{4}\tilde{m}^2\right)\tilde{\rho}^2\right) \tag{36}$$

For $\Lambda_1 \leq 3/4\tilde{m}^2$ (*supersymmetric black holes*), all positive values of $\tilde{\rho}$, $\tilde{M}$ and $\tilde{Z}^2$are admitted. While for $\Lambda_1 > 3/4\tilde{m}^2$, the only allowed maximum radius is $\tilde{\rho}_{\max} = 1\Big/\sqrt{\Lambda_1 - 3/4\tilde{m}^2}$ at which the charge vanishes. For $0 < \tilde{\rho} < \tilde{\rho}_{\max}$, there is an extra positive root given by

$$\sigma = \sqrt{\frac{3}{\Lambda_1 - \frac{3}{4}\tilde{m}^2} - 2\tilde{\rho}^2} - \tilde{\rho} \tag{37}$$

Note that when $0 < \tilde{\rho}^2 < 1/2\left(\Lambda_1 - 3/4\tilde{m}^2\right)^{-1}$, the extra horizon at σ is outside the cold region at $\tilde{\rho}$, while for $1/2\left(\Lambda_1 - 3/4\tilde{m}^2\right)^{-1} < \tilde{\rho}^2 < \left(\Lambda_1 - 3/4\tilde{m}^2\right)^{-1}$, it is inside. Note that the effective temperature at σ is given by [**104**]:

$$T_\sigma^{effective} = \frac{\sigma}{2\pi\left(\sigma^2 + 2\tilde{\rho}\sigma + 3\tilde{\rho}^2\right)}\left(1 - \frac{\tilde{\rho}}{\sigma}\right)\left(1 - \frac{\tilde{\rho}^2}{\sigma^2}\right) \tag{38}$$

When $\tilde{\rho} = 0$, one can easily show that the black hole disappears and find $T_{dS}^{effective} = 1/2\pi\sqrt{1/3\left(\Lambda_1 - 3/4\tilde{m}^2\right)}$ corresponding to a background of effective thermal background in a pure **dS** cosmology [**105**] with $\Lambda_1 > 3/4\tilde{m}^2$. Note that in this case, $\Lambda_1 = 0$ is not permitted unless $\tilde{m}^2 < 0$, or in other words, we admit the possible existence of tachyons (*we will discuss this critical possibility at the end of this paragraph*). Finally, note that in case $\Lambda_1 = -\Lambda_2 = 3/4\tilde{m}^2$, $T_{dS}^{effective} = 0$.

Before passing to the *massless* case, note that for *massive* black hole and for a certain range values of the Reissner-Nordström charge and Schwarzschild mass, the spacetime has three Killing horizons: inner and outer black hole horizons and a positive cosmological constant, that is $\Lambda_1 > 3m^2c^2/4\hbar^2$. The external limit, in which the inner and outer black hole horizons become coincident, occurs when the Schwarzschild mass is less or equal to the absolute value of the Reissner-Nordström charge ($\tilde{M} \leq \left|\tilde{Q}\right|$), with equality in the case $\Lambda_1 = \Lambda_2$, while in the standard case, it corresponds to zero cosmological constant[8]. Following [**106**], one can show that spacetimes containing a number of charge equal to mass black holes with $\Lambda_1 \geq \Lambda_2$, have supercovariantly constant spinors, suggesting that they may be minimum energy states in a positive energy construction[9].

In what follows, the Einstein cosmological constant Λ_1 as well as the Schwarzschild gravitational mass are set equal to zero, that is, $\Lambda_1 = M = 0$. We left with a quantum metric of the following form:

$$ds^2 = c^2\left(1 + \frac{r^2}{\tilde{\lambda}_C^2} + \frac{r_0^2}{r^2}\right)dt^2 - \frac{dr^2}{\left(1 + \frac{r^2}{\tilde{\lambda}_C^2} + \frac{r_0^2}{r^2}\right)} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

[8] A nonzero cosmological constant $\Lambda = -3g^2$ arises in gauged $N = 2$ supergravity, where g is the coupling constant of the gravitino with the $U(1)$ gauge field. If g is real, then $\Lambda < 0$.

[9] Particle production and positive energy theorems for charged **dS** black holes with positive effective cosmological constant will be left to a future work.

$$= c^2\left(\frac{r^4 + r^2\tilde{\lambda}_C^2 + r_0^2\tilde{\lambda}_C^2}{r^2\tilde{\lambda}_C^2}\right)dt^2 - \frac{dr^2}{\left(\frac{r^4+r^2\tilde{\lambda}_C^2+r_0^2\tilde{\lambda}_C^2}{r^2\tilde{\lambda}_C^2}\right)} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{39}$$

$$\overset{m\to 0}{\to} c^2\left(1 + \frac{r_0^2}{r^2}\right)dt^2 - \frac{dr^2}{\left(1+\frac{r_0^2}{r^2}\right)} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{40}$$

which is the classical charged massless black hole described in [**11**]. Here $r_0^2 = q^2G/4\pi\epsilon_0 c^4$. Naturally, such a metric could be obtained starting from *Einstein-Maxwell-Anti-de-Sitter* (**EMAdS**) action[10] [**100**].

In fact, the metric (40) describes a charged massless quantum system and has two naked singularities. Note that $r^4 + r^2\tilde{\lambda}_C^2 + r_0^2\tilde{\lambda}_C^2$ is a quartic algebraic equation for r and has four roots:

$$r_\pm^2 = -\tilde{\lambda}_C^2 \pm \sqrt{\tilde{\lambda}_C^4 - 4r_0^2\tilde{\lambda}_C^2}/2 \tag{41}$$

$$= -2\left(\lambda_C^2 \pm \sqrt{\lambda_C^2\left(\lambda_C^2 - r_0^2\right)}\right) \tag{42}$$

It is evident that both roots are squared negative. One may ask what happens if $r^2$is not allowed to be negative. In this case, r^2 will became greater than zero only if we suppose that, from *hypothetical* point of view, we are dealing with superluminal or sub-eV particles (*for example, hypothetical neutrinos with negative square masses*)[11], than the horizons described in equation (41) or (42) could play significant role. Maybe it will be of interest if black hole evaporation will be responsible of the production of tachyons in the visible universe[12].

To study this critical hypothetical situation, where $r^2 > 0$ (at least one of $r_\pm$), we suppose that $m^2 < 0$, that is $\lambda_C^2 < 0$. If to a certain order, we suppose that $r_0^2 = -\eta\lambda_C^2$ where η is a positive parameter, then $r_\pm^2 = -2\lambda_C^2\left[1 \pm \sqrt{1+\eta}\right]$, and only $r_-^2 > 0$ adding another singular horizons to the problem other than the origin. In this special case, $m^2 = -\eta\left(4\pi\varepsilon_0\hbar^2c^2/q^2G\right) = -\eta M^2 < 0$ where $M = \alpha_{FS}^{-1/2}m_{Pl}$ (*a massive particle*), with $\alpha_{FS} = q^2/4\pi\varepsilon_0\hbar c$ is the fine structure constant and $m_{Pl} = \sqrt{\hbar c/G}$ is the Planck constant. Note that in **GUT's** (*Grand Unification Theories*) the unification mass is given by

[10] By scaling the gauge field A_μ , in order to absorb the prefactors involving the $U(1)$ gauge coupling into the action.

[11] On the experimental discovery that the mass-square of neutrino is negative, the author is referred to [**107**]. In [**108**], based on this discovery, a quantum theory for superluminal neutrino is proposed. But, if neutrino is a tachyon, then all the weak interaction in which neutrino participates need to be restudied. See also [**109**].

[12] We believe that if tachyons are emitted by a black hole, the quantum theory will not be suitable in explaining the physics of black hole, or at least must be defined in another way in this work; whatever is the case, we will not deal with this problem in this work, neither dealing with the possibility of evaporation of black holes into tachyons, we are mentioning this case from phenomenological point of view, not only.

$m_{GUT} \approx \alpha_{FS}^{1/2} m_{Pl}$, characterizing gravito-electroweak unification scale [**110,111**]. This seems interesting to have in function of the **GUT** mass scale, $m_{GUT} \approx \alpha_{FS}^{-1} m_{Pl}$. In addition to the gravitons (*as well as the other particles*) are radiated with the Hawking temperature from the black hole [**112,113**], the implementation of our quantum mass into this theory is an important feature and could have important impacts on black hole and massive remnants. Some quantum gravity researchers claim that the Hawking process terminates with the production of a large, massive remnant which carries all of the information missing from the outgoing radiation. Whether the remnant subsequently decays or whether it is long lived, the information stored in it becomes accessible [**114,115,116**]. Electroweak gauge bosons have masses of the order of $10^2 GeV(/c^2)$ while masses of additional bosons involved in gravito-electroweak unification are expected to be still higher. These are at least eleven orders of magnitude higher than sub-eV range indications for neutrino masses. If under these circumstances we suspect that the sub-eV particles are created in a spacetime where gravitational effects of massive gauge bosons may become important, than one may ask on the nature of the spacetime group around a gravito-electroweak vertex [**117**]. In this later reference, the authors modeled this spacetime as **dS** background and find that sub-eV particles may carry a negative mass square of the order of $-3(3/8\pi^2)(M_{GUT}^4/m_{Pl}^2)$.

6. Conclusions

The study of extreme and nearly extreme black holes, in particular the massless **BTZ** black hole background as well as supersymmetric massless black holes has recently been increased, mainly due to the link with one of the *quantum* puzzle which is still unsolved: the issue related to the statistical physical interpretation of the Bekenstein-Hawking entropy. Explaining and understand physically this point is recognized as an essential hallmark to complete the quantum theory of gravity.

In this work, we have used the non-minimal coupling to implement the quantum Compton wavelength naturally in the Einstein field equations. At this level, the field equations are viewed with two cosmological constant or an effective one $\Lambda = \Lambda_1 + \Lambda_2$ where Λ_1 is the standard, that is Einstein's cosmological constant and Λ_2 (*which is proportional to the inverse square of the Compton wavelength*) is our quantum second negative cosmological constant. We have been also interested on the topology where $\Lambda_1 = 0$. We have then constructed a quantum massless **AdS** black hole singular free, regular and globally homeomorphic to $AdS_2 \times S_2$.

For *massive* **SdS** black hole, there are two radius corresponding to black horizon (r_+) and cosmological horizon (r_c). For $\Lambda_1 > 3m^2c^2/4\hbar^2$, as the Schwarzschild mass increases, the black hole radius r_+ increases and the cosmological horizon r_c decreases monotonically. For maximum value of M, $r_+ = r_c = \sqrt{1/\Lambda_1 + \Lambda_2}$, which is the biggest black hole that can be formed in de Sitter space. In case $\Lambda_1 < 3m^2c^2/4\hbar^2$, we have **SAdS**. For a very small black hole, and by considering a system consisting of initially well-separated matter and black hole, the first law of thermodynamics gives approximately the temperature $T = 1/4\pi\zeta r_c$ providing that $\Lambda_1 > 3m^2c^2/4\hbar^2$. The entropy is given by $S = \pi r_+^2 = 1/4A$ where A is the area of the event horizon. This is the same form of entropy s in asymptotically flat space.

We showed also that quantum **AdS** free-charged black hole behaves as a relativistic harmonic oscillator. While for charged black holes, interesting features arise. First, for zero classical Einstein's cosmological constant ($\Lambda_1 = 0$) and negative *quantum* one ($\Lambda_2 = -3/4\tilde{m}^2$), the square of the scalar curvature $R^2 > 0$. For $\Lambda_1 > 3/4\tilde{m}^2$, there exist an additional event horizon$\approx \sqrt{3/\Lambda_1 + \Lambda_2}$ constituting the "*outer edge*" of the **dS** universe in the given coordinate system. In asymptotically flat space-time, in particular for $\tilde{Z}^2 < \tilde{M}^2$, there exists two horizons at radii $\tilde{\rho}_\pm = \tilde{M} \pm \sqrt{\tilde{M}^2 - \tilde{Z}^2}$. Taking $\tilde{Z}^2 = \Lambda_1 = 0$, the Hawking temperature corresponds to an uncharged cosmological **AdS** black hole, while in the standard case, it corresponds to a **dS** one. Horizons with zero Hawking temperature are found to be located at double roots of $g_{00}(\tilde{\rho})$). When such a double roots exists, and in particular for $\Lambda_1 > 3/4\tilde{m}^2$, the only allowed maximum radius is $\tilde{\rho}_{\max} = 1\Big/\sqrt{\Lambda_1 - 3/4\tilde{m}^2}$ at which the charge vanishes. When $\tilde{\rho} = 0$, one can easily show that the black hole disappears with an effective **dS** temperature given by $T_{dS}^{effective} = 1/2\pi\sqrt{1/3\left(\Lambda_1 - 3/4\tilde{m}^2\right)}$ corresponding to a background of effective thermal background in a pure **dS** cosmology. $\Lambda_1 = 0$ in this case is not permitted unless we admit the possible existence of tachyons, that is $\tilde{m}^2 < 0$. Phase transitions of thermal radiation in **SAdS** will be dealt in future work. Another important question we would like to answer in another work is how the presence of the two cosmological constants Λ_1 and Λ_2 reflects itself in properties of the motion of test particles and photons, the photon escape cones and the embedding diagrams [**118**].

For *massless* black hole , an interesting *phenomenological* feature arises when we admit the presence of other than $r = 0$ horizon. In this case, we are argued to accept the possible production of sub-eV particles, that is, tachyons with negative mass square of the same order of the square of gravito-electroweak unification scale masses, pushing us to ask about the validity of the quantum theory and whether or not this later is really suitable to explain the mystery of quantum gravity black holes physics, and about the nature of physics that could describe their internal structures. Thus, a more careful study of their backgrounds, as well as their holographic properties will be crucial in order to shed light on their real quantum nature.

One certainly needs understand better all these issues (*in addition to string-black holes interactions, foams, holographic properties,...*) and we hope to return to them in further publications.

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