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Duality and a Renormalization Scheme for Einsteinian Gravity as a Fix Point Within a Gravitational Gauge Framework

Eckehard W. Mielke *

Departamento de Física, Universidad Autónoma Metropolitana–Iztapalapa, Apartado Postal 55-534, C.P. 09340, México, D.F., MEXICO

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Abstract: A general scheme for a *field redefinition* (FR) of the coframe and the connection is developed. Within a Yang–Mills type gauge dynamics of gravity, configurations with double dual curvature induced by a θ-type Chern-Simons terms as generating function reside on an effective Einsteinian background. The effect of the FR on the renormalization and the relation of gravity to effective string models is studied. One encounters a *duality of weak and strong couplings* of Einsteinian and renormalizable Yang–Mills type gravity as well as an induced cosmological constant of the Anti–de Sitter space.

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1. Introduction

The *duality* of electric and magnetic fields in Maxwell's theory was already known to Von Laue [91] and Silberstein [89]. Later it was generalized to the symmetry of *duality rotations* by Rainich [87] and developed further in *geometrodynamics* by Misner and Wheeler [74], cf. [61]. More recently, Motonen and Olive [76, 82] noted that then also a duality of the strong-weak coupling regime of gauge fields is generated, the so-called *S–duality.* In the context of magnetic monopoles it plays nowadays a predominant role in M–theory [18, 96].

We are going to apply this to a Yang–Mills–type formulation of gravitational interactions, regarding it as a *field redefinition* [38, 39, 66]. In general, not only the energy–momentum content of matter, but also its spin couples to a dynamical geometry with

* ekke@xanum.uam.mx

translational and Lorentz-rotational curvature [32, 34]. This general framework encompasses the Einstein–Cartan (EC) theory as an important (but from the variational point of view degenerate) subcase which is macroscopically indistinguishable from Einstein's theory of general relativity (GR).

2. Field Redefinition Scheme

In perturbative quantum gravity [1], there arise counterterms ΔV of higher order in the curvature. According to 't Hooft [38, 39], these terms can be simulated, already on the classical level, by a *field redefinition* (FR)

$$g_{ij} \to \widetilde{g}_{ij} = g_{ij} + aRic_{ij} + bg_{ij}\, Ric_k{}^k \tag{2.1}$$

of the metric in the first order approximation. In exterior form notation, the symmetric Ricci tensor is the holonomic version of the zero-form $Ric_{\alpha\beta} := {}^*(R_{(\alpha}{}^{\delta} \wedge \eta_{\delta|\beta)})$. In a gauge framework based on the Poincaré group and summarized in the Appendix, the independent variables are the one-forms ϑ^α and $\Gamma^{\alpha\beta}$. Then a generally nonlinear FR of these basic variables are dictated by the appropriate form degree and the correct physical dimension:

$$\vartheta^\alpha \quad \to \quad \widetilde{\vartheta}^\alpha := \vartheta^\alpha + \ell^2\, e_\beta\rfloor {}^*H^{\alpha\beta}\,, \tag{2.2}$$

$$\Gamma_\alpha{}^\beta \quad \to \quad \widetilde{\Gamma}_\alpha{}^\beta := \Gamma_\alpha{}^\beta + \ell^2\, e_\alpha\rfloor {}^*H^\beta\,. \tag{2.3}$$

Here the field momenta H_α and $H_{\alpha\beta}$ are understood as arising from a generating n–form G as part of some effective gauge Lagrangian $V_{\rm eff}$ which includes the counterterms from the searched-for renormalization. Observe also that a *fundamental length* ℓ squared necessarily occurs for dimensional reasons. These *deformations* of the gauge potentials can also be viewed as *nonlinear prolongations* [6].

As a consequence, the spacetime metric

$$g = g_{ij}\, dx^i \otimes dx^j = o_{\alpha\beta}\, \vartheta^\alpha \otimes \vartheta^\beta\,. \tag{2.4}$$

gets redefined by

$$g \quad \to \quad \widetilde{g} = g + 2\ell^2 o_{\alpha\beta}\vartheta^\alpha \otimes e_\mu\rfloor {}^*H^{\beta\mu} + \ell^4 o_{\alpha\beta} e_\mu\rfloor {}^*H^{\alpha\mu} \otimes e_\nu\rfloor {}^*H^{\beta\nu}\,, \tag{2.5}$$

i.e. by curvature excitations[1] up to quadratic order. This generalizes 't Hooft's ansatz (2.1) for the metric, used there in an attempt at perturbative renormalization of GR on a Riemannian background. For a general counterterm ΔV in the effective gauge Lagrangian $V_{\rm eff}$, our geometrical variables become redefined according to the "intertwining relations" (2.2,2.3) via the corresponding $H_{\rm eff}$.

[1] For nonvanishing non-metricity $Q_{\alpha\beta} := -Dg_{\alpha\beta}$, there would be an additional FR of the anholonomic metric via $g_{\alpha\beta} \quad \to \quad \widetilde{g}_{\alpha\beta} = g_{\alpha\beta} + {}^*m_{\alpha\beta}$ where $m_{\alpha\beta} := -\partial G/\partial g_{\alpha\beta}$ is an n–form, cf. Eq. (3.11.13) of Ref. [34].

The dual two-form and the volume four–form are deformed according to

$$\eta_{\alpha\beta} \quad \rightarrow \quad \widetilde{\eta}_{\alpha\beta} = \eta_{\alpha\beta} + \ell^2 \eta_{\alpha\beta\gamma} \wedge e_\mu \rfloor {}^*H^{\mu\gamma} + \frac{\ell^4}{2} \eta_{\alpha\beta\gamma\delta} (e_\mu \rfloor {}^*H^{\gamma\mu}) \wedge (e_\nu \rfloor {}^*H^{\delta\nu}) \,, \tag{2.6}$$

and

$$\begin{aligned} \eta \quad \rightarrow \quad \widetilde{\eta} &= \eta + \ell^2 \eta_{\alpha\beta} \wedge {}^*H^{\alpha\beta} + \frac{\ell^4}{2} \eta_{\alpha\beta} (e_\mu \rfloor {}^*H^{\alpha\mu}) \wedge (e_\nu \rfloor {}^*H^{\beta\nu}) + \cdots \\ &+ \frac{\ell^8}{4!} \eta_{\alpha\beta\gamma\delta} (e_\mu \rfloor {}^*H^{\alpha\mu}) \wedge (e_\nu \rfloor {}^*H^{\beta\nu}) \wedge (e_\rho \rfloor {}^*H^{\gamma\rho}) \wedge (e_\omega \rfloor {}^*H^{\delta\omega}) \,, \end{aligned} \tag{2.7}$$

respectively.

In our dynamical approach, the $(n-2)$–forms H_α and $H_{\alpha\beta}$ will be gauge field momenta canonically conjugated to the coframe and the Lorentz connection, respectively. Due to the *semidirect product* structure of the Poincaré group $P := R^4 \otimes SO(1,3)$, the gauge field momenta contribute to the gauge potentials via $H^{\alpha\beta} \rightarrow \vartheta^\alpha$ and $H^\alpha \rightarrow \Gamma^{\alpha\beta}$ in the field redefinition (2.2,2.3) just in an *intertwined* manner.

In the four–dimensional gauge theory, a Hodge star for gauge field momenta H can be dismissed as it is done in Ref. [66], but in dimension $n \neq 4$ it is necessary to use ${}^*H$ in order obtain the correct form degree. In the FR (2.3) of the connection we could have included, similarly as for Yang–Mills fields, the term $\ell^2\, {}^*DH_\alpha{}^\beta$. However, 'on shell', i.e. when the vacuum field equation (5.18) is satisfied, this is equivalent to ${}^*\left(H^\beta \wedge \vartheta_\alpha\right) \equiv e_\alpha \rfloor {}^*H^\beta$ due to the identity (3.7.13) of Ref. [34]. In the FR (2.2) of the coframe, the same situation arises, with the modification that the 'on shell' term ${}^*DH^\alpha \cong {}^*E^\alpha$ is second order in the field strength and therefore equivalent to a higher order generation functional G. When coupling to matter, FRs have to be taken with care because they may induce violations of the macroscopic principle of equivalence, cf. Brans [11].

2.1 Legendre transformation

For exhibiting physically equivalent gauge field Lagrangians via a Legendre transformation, let us proceed from the Hilbert–Einstein of general relativity (GR) or the Einstein–Cartan (EC) Lagrangian

$$V_{\mathrm{EC}} = -\frac{1}{2\ell^2} R^{\alpha\beta} \wedge \eta_{\alpha\beta} \tag{2.8}$$

as an example [95].

If we compare this with the more general Lagrangian

$$\widetilde{V} = -\sum_{k=0}^{K} (1/2k)\, R^{\alpha\beta} \wedge \overset{(2k)}{H}{}_{\alpha\beta} + V_\theta \,, \tag{2.9}$$

which is quadratic, quartic, etc. in the curvature. The first term in this expansion corresponds to the Stephenson-Kilmister-Yang (SKY) Lagrangian quadratic in the curvature [94, 36, 46, 98, 58, 97]. The *gauge field momentum* $\widetilde{H}_{\alpha\beta} := -\partial \widetilde{V}/\partial R^{\alpha\beta}$ can be expanded as $\widetilde{H}_{\alpha\beta} = \overset{(2)}{H}{}_{\alpha\beta} + \overset{(4)}{H}{}_{\alpha\beta} + \cdots$. For the time being, the field momentum $\widetilde{H}_\alpha := -\partial \widetilde{V}/\partial T^\alpha$

conjugated to the torsion $T^\alpha := D\vartheta^\alpha$ is put to zero. From the field equations (5.17, 5.18) of PG theory, we can then infer the resulting *Yang–Mills type field equations*

$$-\widetilde{E}_\alpha := -e_\alpha \rfloor \widetilde{V} - (e_\alpha \rfloor R^{\beta\gamma}) \wedge \widetilde{H}_{\beta\gamma} = \Sigma_\alpha \,, \tag{2.10}$$

$$D\widetilde{H}_{\alpha\beta} = \tau_{\alpha\beta} \,. \tag{2.11}$$

However, the *Legendre transformation* [42, 66]

$$\widetilde{V} \;\to\; V = -\frac{1}{2}\left(R^{\alpha\beta} \wedge \widetilde{H}_{\alpha\beta} - \widetilde{V}\right) \tag{2.12}$$

provides a physically equivalent gravitational Lagrangian V. (The overall factor 1/2 is chosen so as to render the EC Lagrangian invariant.) The new rotational gauge field momenta

$$H_{\alpha\beta} := -\frac{\partial V}{\partial R^{\alpha\beta}} = \widetilde{H}_{\alpha\beta} + \frac{1}{2} R^{\mu\nu} \wedge (\partial \widetilde{H}_{\mu\nu}/\partial R^{\alpha\beta}) \,, \tag{2.13}$$

will depend on the *Hessian*

$$\widetilde{H}_{\alpha\beta\mu\nu} := \frac{\partial^2 \widetilde{V}}{\partial R^{\mu\nu} \partial R^{\alpha\beta}} = -\frac{\partial \widetilde{H}_{\mu\nu}}{\partial R^{\alpha\beta}} \tag{2.14}$$

of the Lagrangian $\widetilde{V}$ we started with.[2]

2.2 Constant Hessian and S–duality

As the first illustrative case let us first consider a *constant* Hessian, i.e.

$$\widetilde{H}_{\alpha\beta\mu\nu} = 2\theta_{\mathrm{L}} g_{[\alpha|\mu} g_{\beta]\nu} + \theta^\star_{\mathrm{L}}\, \eta_{\alpha\beta\mu\nu} \tag{2.15}$$

arising from the curvature type θ terms in the topological term V_θ. Then from (2.13) the following form for the new rotational field momenta can be inferred

$$H_{\alpha\beta} = \widetilde{H}_{\alpha\beta} - \theta_{\mathrm{L}} R_{\alpha\beta} - \theta^\star_{\mathrm{L}} R^{(\star)}_{\alpha\beta} \,, \tag{2.16}$$

where we disregard for the moment a kernel discussed below in (2.19). Note that the θ-terms can be regarded as induced via the boundary terms (5.15) of Pontrjagin and

[2] Capovilla et al. [12] presented a reformulation of GR in which a gauge potential and an arbitrary scalar density $\sigma = \sqrt{|\det g_{\mu\nu}|}$ (but no metric) occur as dynamical variables. Then the Hodge dual is constructed via $\eta_{\alpha\beta\gamma\delta} := \sigma\, \epsilon_{\alpha\beta\gamma\delta}$, i.e. from the Levi–Civita symbol multiplied by this scalar density. Following Jakubiec and Kijowski [42], we have pointed out [66] that this reformulation as well as Ashtekar's complex variables [2, 3] can be interpreted as a FR applied to the EC Lagrangian. By translating the $SO(3,C)$ formulation of Ref. [12] into our formalism, the equivalent of $\overset{(4)}{H}_{\alpha\beta} = R^{\gamma\delta} \wedge {}^*(R_{\alpha\gamma} \wedge R_{\beta\delta})$ is the only nonvanishing term. By varying $\widetilde{V}$ with respect to $\Gamma^{\alpha\beta}$ and σ, the *vacuum* field equation $D\widetilde{H}_{\alpha\beta} = 0$ and the energy–momentum trace $(\vartheta^\alpha \wedge \widetilde{E}_\alpha)/\sigma = (4\widetilde{V} + 2R^{\alpha\beta} \wedge \widetilde{H}_{\alpha\beta})/\sigma = 0$ are found. If we insert this $\overset{(4)}{H}_{\alpha\beta}$ into the metric, the curvature plays the role of a 'cubic root' of the deformed metric (2.5), similarly as in the chiral alternative of 't Hooft [40]. In our scheme. however, also order six terms will arise.

Euler, respectively. Using the decomposition (5.12) into self– or antiselfdual fields with respect to the Lie dual, this can be resolved for the curvature as

$$R^{(\pm)}_{\alpha\beta} = \frac{1}{(\theta_{\rm L} \pm \theta^\star_{\rm L})} \left(\widetilde{H}^{(\pm)}_{\alpha\beta} - H^{(\pm)}_{\alpha\beta} \right) \tag{2.17}$$

with a coupling constant which is *inverse* to those of the θ terms. Then upon the Legendre transformation (2.12), the new Lagrangian can be rewritten as

$$V = -\frac{1}{2(\theta_{\rm L} \pm \theta^\star_{\rm L})} \left(\widetilde{H}^{(\pm)}_{\alpha\beta} - H^{(\pm)}_{\alpha\beta} \right) \wedge \widetilde{H}^{\alpha\beta} + \frac{1}{2}\widetilde{V}\,. \tag{2.18}$$

Thereby the originally *weak* coupling to the θ terms is converted to a *strong* coupling regime $1/\theta$ for the field momenta and vice versa. This so-called *S–duality* of strong and weak coupling was first noted by Motonen and Olive [76] in the contex of magnetic monopoles and plays nowadays a prominant role in M-theory [18, 96]. In the context of non-Abelian Yang–Mills theories, a related equivalence with respect to S–duality was first encountered in Refs. [24, 75] and then, in a particular case, applied [26] to the MacDowell–Mansouri gauge theory of gravity .

2.3 Vanishing Hessian: GR as a stable fix point

Since EC theory may arise from different higher order models, we have an infinite ambiguity in such a "renormalization" program, cf. Kaku [44], p. 210. However, we can improve this by showing that EC theory is a *stable* fix point of the quadratic SKY gravity, e.g. For a *fixed point* of the transformation (2.12), the Hessian $\widetilde{H}_{\alpha\beta\mu\nu}$ obviously has to vanish. This condition, i.e. $\partial\widetilde{H}_{\mu\nu}/\partial R^{\alpha\beta} = 0$, can be readily solved. If parity violating terms such as $\theta_{\rm T}\, R^{\alpha\beta} \wedge \vartheta_\alpha \wedge \vartheta_\beta$ arising from the Nieh-Yan term (5.7) are admitted, then we obtain the relation

$$\widetilde{\eta}_{\alpha\beta} := \frac{\theta^*_{\rm T}}{2}\eta_{\alpha\beta} - \frac{\theta_{\rm T}}{2}\,\vartheta_\alpha \wedge \vartheta_\beta - \ell^2\,\widetilde{H}_{\alpha\beta} = 0 \tag{2.19}$$

which can be regarded as a singular FR derived from (2.2), but arising from a different effective[3] Lagrangian $\widetilde{V}$.

Accordingly $\eta_{\alpha\beta}$ and $\widetilde{H}_{\alpha\beta}$ interchange their role as generalized coordinates and momenta, respectively. If we had started from $\widetilde{V} = V_{\rm EC}$ then we would be led back to $V = V_{\rm EC}$ for the choice $\theta^*_{\rm T} = 1$ and $\theta_{\rm T} = 0$. In the case $\theta^*_{\rm T} = 1$ and $\theta_{\rm T} = i$ this leads to a *chiral* formulation of gravity [65]. Thus, the EC Lagrangian or its chiral version remains as a *"stable"* Lagrangian under FR, provide we embbed it in a class of gravity Lagrangians for which $V_{\rm EC}$ is located at some local minimum.

Our gauge framework clearly exhibits the coupling to fundamental matter, such as to the Dirac field. If we reinsert (2.19) into (2.11), we recover the algebraic Cartan type equation $\eta_{\alpha\beta\gamma} \wedge T^\gamma = 2\ell^2\,\tau_{\alpha\beta}$. In the Dirac case, this implies a nonvanishing *axial*

[3] In D=11 supergravity, a similar relation holds after compactification for the 7-volume form on S^7, i.e. $\eta_7 \approx H = -\partial V/\partial dB$, where B is the Kalb–Ramond three–form, cf. [18].

torsion. However, as already stressed by 't Hooft [39], a FR of the *coframe* may ruin the nice features of the Dirac Lagrangian which, in GR and its RC extensions, has to be formulated in terms of ϑ^α in a *multiplicative* way: Dangerous derivative couplings must be avoided in the Dirac equation and the positivity of energy needs to asured during this procedure [90].

In the transformation to Ashtekar's complex variables, the coframe is kept fixed, whereas the connection is subjected to the *complex* FR $\Gamma_\alpha{}^\beta \quad\rightarrow\quad \overset{(\pm)}{\Gamma}{}_\alpha{}^\beta := \Gamma_\alpha{}^\beta \mp (i\ell^2/2)\, e_\alpha\rfloor \overset{(\mp)}{H}{}^\beta$, induced by the translational Chern–Simons term $idC_{\rm TT}$, cf. [62, 63]. The resulting Sen type connection still couples *minimally* to the Dirac field, but poses the issue of *reality conditions*, cf. [69].

2.4 GR from effective strings

A more general situation arises, when we consider the tree–level effective action, corresponding to the lowest order in the string loop expansion, in the physical Einstein frame: According to Damour and Vilenkin [14], the following nonlinear terms arise

$$\begin{aligned} V_{\rm Stringeff} &= \left(\frac{1}{\alpha'}\hat{R} + \hat{R}^2 + (\alpha')\hat{R}^3 + (\alpha')^2\hat{R}^4 + \cdots\right)\eta \\ &= \frac{1}{(\alpha')^2}\sum_{n=1}^{\infty}(\alpha'\hat{R})^n\eta = \frac{\hat{R}/\alpha'}{1-\alpha'\hat{R}}\eta\,, \end{aligned} \tag{2.20}$$

where α' is the slope parameter and $\hat{R}^n$ stands in for generic higher–order curvature invariants.

For $|\alpha'\hat{R}| < 1$, the formal summation to a *geometric series* is inspired by the *nonlinear graviton* construction of Penrose [85]. To be justified, we need, for simplicity, to identify $\hat{R} = R = {}^*(R^{\alpha\beta}\wedge\eta_{\alpha\beta})$ with the curvature scalar. Then the Lorentz-rotational field momentum (5.19) reduces to $\widetilde{H}_{\alpha\beta} := -(\partial\widetilde{V}/\partial R)(\partial R/\partial R^{\alpha\beta}) = -\eta_{\alpha\beta}(\partial V/\partial R)$, and the Legendre transformation (2.12) simplifies to

$$\begin{aligned} \widetilde{V} \;\rightarrow\; V = \frac{1}{2}R\frac{\partial\widetilde{V}}{\partial R} + \frac{1}{2}\widetilde{V} &= \frac{R/\alpha'}{1-\alpha' R}\eta + \frac{R^2}{2(1-\alpha' R)^2}\eta \\ &\simeq \frac{1}{\alpha'}R\eta + \frac{1}{2}R^2\,. \end{aligned} \tag{2.21}$$

Again the Lagrangian truncated at $\alpha' \rightarrow 0$ corresponds to the perturbatively renormalizable quadratic model of Stelle [93].

3. Field Redefinition Induced by Theta Terms

In the gauge framework with torsion, the most general *quadratic* Lagrangian[35, 32, 81] reads

$$
\begin{aligned}
V_{\rm QPG} &= \frac{\Lambda}{\ell^2}\,\eta + \frac{a_0}{4\ell^2} R^{\alpha\beta}\wedge\eta_{\alpha\beta} - \frac{1}{2} T^\alpha \wedge H_\alpha - \frac{1}{2} R^{\alpha\beta}\wedge H_{\alpha\beta} + V_\theta \\
H_\alpha &:= -\frac{1}{\ell^2}\,{}^*\left(\sum_{M=1}^{3} a_{(M)}\,{}^{(M)}T_\alpha\right), \quad H_{\alpha\beta} := -\frac{a_0}{2\ell^2}\,\eta_{\alpha\beta} - \frac{1}{g^2}\,{}^*\left(\sum_{N=1}^{6} b_{(N)}\,{}^{(N)}R_{\alpha\beta}\right)
\end{aligned}
\tag{3.1}
$$

The dimensionless coupling constant g and the fundamental length ℓ fix the relative strength of the rotational and translational interaction parts of the gravitational Lagrangian V. In the field momenta, each of the three irreducible torsion and six irreducible curvature pieces contribute to the Lagrangian[4] with an individual weight $a_{(M)}$ and $b_{(N)}$, respectively.

If we resolve the condition of *constant Hessian* for the allowed form of the rotational field momenta , we obtain the *generalized double duality* ansatz (DD)

$$
H_{\alpha\beta}(**) = \theta_{\rm L}\, R_{\alpha\beta} + \theta_{\rm L}^\star\, R_{\alpha\beta}^{(\star)} + \frac{\theta_{\rm T}^*}{2\ell^2}\,\eta_{\alpha\beta} - \frac{\theta_{\rm T}}{2\ell^2}\vartheta_\alpha\wedge\vartheta_\beta\,, \tag{3.2}
$$

where $\theta_{\rm T}$, $\theta_{\rm T}^*$, $\theta_{\rm L}$, and $\theta_{\rm L}^\star$ are dimensionless constants which can be related to the individual coupling constants in the θ–type boundary term (5.15). It has been demonstrated in much detail elsewhere [58, 59, 61, 72, 99] that the DD ansatz maps the second field equation (5.18) into the second Bianchi identity (5.6), provided the translational gauge field momenta fulfill certain algebraic conditions. By inserting the duality ansatz[5] into the second field equation (5.18), these may be derived from

$$
\frac{\theta_{\rm T}^*}{2\ell^2}\eta_{\alpha\beta\gamma}T^\gamma - \vartheta_{[\beta}\wedge H_{\alpha]}(**) + \frac{\theta_{\rm T}}{\ell^2}\vartheta_{[\beta}\wedge T_{\alpha]} = \tau_{\alpha\beta}\,. \tag{3.3}
$$

[4] The propagating modes and particle content of this Lagrangian were investigated by Sezgin and van Nieuwenhuizen[88]. A subclass of Lagrangians survived their selection criteria motivated by quantum field theoretical considerations, such as ghost-freeness and positive energy of the physical modes. By performing a mode decomposition based on a *flat* Minkowskian background, Kuhfuß and Nitsch [49] found a three–parameter class of unitary PG Lagrangians (see [32] and further References). therein. However, there may arise problems with the Cauchy formulation, shock waves [51], and positivity of the gravitational energy, see Hecht et al. [30, 31] as well as [27] for a review. A more general class has been employed [19] in a FR, where axial torsion gets identified with a dynamical axion field.

[5] Instanton solutions of the Stepenson-Kilmister-Yang (SKY) theory of gravity were already 1981 classified [58] with such an ansatz simplified by the choice $\theta_{\rm T} = \theta_{\rm T}^* = \theta_{\rm L} = \theta_{\rm L}^\star = 0$; much earlier than, e.g., García-Compeán et al. [25] in the context of S-duality. (For an extension to metric-affine theory, see [97].) In the wider framework of quadratic gravitational gauge models this 1982 ansatz of Baekler et al. [7] for $\theta_{\rm L} = 0$ induces for purely imaginary $Im(\theta_{\rm T}) \neq 0$ *complex* dual variables similar to those found later by Ashtekar [2] and Minkowski [73]. Recently, Soo [92] 'recovered' a more specialized version of our earlier DD ansatz. The modified self-dual action of Barbero [9] corresponds to a real $\theta_{\rm T}$, but necessarily faces the problem of CP violation, cf. [71]. The same problem arises in the so-called 'Immirzi ambiguity' [41], which is generated [37] by a part of the Nieh-Yan term (5.7). Contrary to the statement of Gambini et al. [23], this translational $\theta_{\rm T}$ is a total divergence in RC spacetime.

For spinless matter (cf. [59] for the general case), the translational momentum $H_\alpha(**)$ subject to the constraint (3.2) takes the form

$$H_\alpha(**) = -\frac{\theta_{\rm T}^*}{\ell^2}\,{}^*\Big[T_\alpha - \vartheta_\alpha\wedge(e_\beta\rfloor T^\beta) - \frac{1}{2}\,e_\alpha\rfloor(T^\beta\wedge\vartheta_\beta)\Big] + \frac{\theta_{\rm T}}{\ell^2}T_\alpha$$
$$= -\frac{\theta_{\rm T}^*}{2\ell^2}\,K^{\beta\gamma}\wedge\eta_{\alpha\beta\gamma} + \frac{\theta_{\rm T}}{\ell^2}T_\alpha = \frac{\theta_{\rm T}^*}{\ell^2}\,K_\alpha^{(\star)} + \frac{\theta_{\rm T}}{\ell^2}T_\alpha. \qquad (3.4)$$

As a further consequence, the first field equation (5.17) reduces to the Einstein equation

$$\frac{\theta_{\rm T}^*}{2}R^{\{\}\beta\gamma}\wedge\eta_{\alpha\beta\gamma} - \Lambda_{\rm eff}\,\eta_\alpha = \ell^2\,\widehat{\Sigma}_\alpha \qquad (3.5)$$

for the Riemannian background with an *effective* “cosmological” constant

$$\Lambda_{\rm eff} = \Lambda - \Lambda_\theta\,, \qquad \Lambda_\theta = \frac{3(\theta_{\rm T}^* + a_0)^2}{2\ell^2(\theta_{\rm L} + \theta_{\rm L}^\star + b_6/g^2)} \qquad (3.6)$$

of microscopic origin [59, 61, 8, 54, 55]. Observe that the ‘bare’ cosmological constant Λ gets subtractively *renormalized* by a term induced by the Lorentz rotational boundary terms. For $b_6 = 0$, this persists even in the weak coupling limit $g \to 0$.

By inserting (3.2) and (3.4) into (3.1), the same can be obtained on the Lagrangian level. Since the torsion terms drop out due to the Nieh-Yan relation (5.7) and the teleparallelism identity (5.14), we are left with an *effective* Hilbert–Einstein Lagrangian

$$V_{\rm eff} = -\frac{\theta_{\rm T}^*}{2\ell^2}R^{\{\}\alpha\beta}\wedge\eta_{\alpha\beta} + \Lambda_{\rm eff}\,\eta\,. \qquad (3.7)$$

This result complies with that of Refs.[59, 8], because the topological boundary terms have already been included in the quadratic Lagrangian (3.1) we started with. In order to attain macroscopic correspondence, the coupling of the effective Einstein equation (3.5) to the symmetrized [67] energy–momentum current $\widehat{\Sigma}_\alpha$ of matter requires $\theta^*_{\rm Tphys.} = 1$ for consistency.

What are the consequences of the duality ansatz for a *general* gravitational gauge model in terms of the field redefinition of the basic fields, the coframe and connection? Since $e_\beta\rfloor\eta^{\alpha\beta} = 0$, we find

$$\widetilde{\vartheta}^\alpha = (1 + \frac{3}{2}\theta_{\rm T}^*)\vartheta^\alpha - \theta_{\rm L}\ell^2\,e_\beta\rfloor{}^*R^{\alpha\beta} - \theta_{\rm L}^\star\ell^2\,e_\beta\rfloor{}^*R^{\alpha\beta(\star)}\,, \qquad (3.8)$$

$$\widetilde{\Gamma}_\alpha{}^\beta = \Gamma_\alpha{}^\beta + \frac{\theta_{\rm T}^*}{2}e_\alpha\rfloor{}^*K^{\beta(\star)} + \theta_{\rm T}\,e_\alpha\rfloor{}^*T^\beta = \Gamma_\alpha^{\{\}\beta} - K_\alpha{}^\beta + \frac{\theta_{\rm T}^*}{2}e_\alpha\rfloor{}^*K^{\beta(\star)} + \theta_{\rm T}\,e_\alpha\rfloor{}^*T^\beta\,. \quad (3.9)$$

Besides a different normalization, a curvature piece and a double dual one related to the Euler four–form (5.13) deformes the coframe. In the special case $\theta_{\rm T}^* = -2/3$, the coframe (or holonomic metric) arises as a concept derived from the curvature, thus leading to an Eddington type theory [20]. On the other hand, in the deformed connection occurs only a reshuffling of the contortional pieces such that the Riemannian connection stays invariant. It is interesting to note that then the volume four–form (2.7), will change for the simplifying choice $\theta_{\rm L} = 0$ to

$$\widetilde{\eta} = (\theta_{\rm L}^{\star})^4 \frac{\ell^8}{4!}\, \eta_{\alpha\beta\gamma\delta} e_\mu \rfloor {}^*R^{\alpha\mu(\star)} \wedge \cdots \wedge e_\mu \rfloor {}^*R^{\delta\mu(\star)}\,, \qquad (3.10)$$

which for $\ell = \ell_{\rm Planck}$ is of the order 10^{-256} times $\theta_{\rm L}^{\star 4}|R|^4$. Do we have here some clue for a macroscopically tiny effective cosmological term $\Lambda\widetilde{\eta}$ induced by the Lorentz-rotational $\theta_{\rm L}^{\star}$–term? Or does this correspond to the sought-for "dark energy" induced by Chern-Simons terms?

4. Duality of Weak and Strong Coupling Limit

Can we use this FR scheme in order to start from a general PG Lagrangian which avoids Cauchy, ghost mode, and renormalization problems and end up after renormalization with an effective Einstein equation at the *macroscopic* level?

In order to probe the virtue of our construction on a more simple model, let us compare the Hilbert–Einstein or EC Lagrangian with the Lagrangian

$$\widetilde{V}_{\rm SKY} = -\frac{1}{2g^2} R^{\alpha\beta} \wedge {}^*R_{\alpha\beta} + \frac{1}{2}\theta_{\rm L}^{\star}\, R^{\alpha\beta} \wedge R^{(\star)}_{\alpha\beta} \qquad (4.1)$$

of SKY gravity supplemented by the Euler term, where $\widetilde{H}_{\alpha\beta} = {}^*R_{\alpha\beta}/g^2 - \theta_{\rm L}^{\star} R^{(\star)}_{\alpha\beta}$. From the work of Stelle [93] we know that this curvature squared gravity is perturbatively *renormalizable* but plagued with ghost [50].

Via the singular FR

$$\eta_{\alpha\beta} \quad \to \quad \widetilde{\eta}_{\alpha\beta} := \eta_{\alpha\beta} - \frac{\ell^2}{g^2}\, {}^*R_{\alpha\beta} + \theta_{\rm L}^{\star}\ell^2\, R^{(\star)}_{\alpha\beta} = 0\,. \qquad (4.2)$$

we retain from the Legendre transformation (2.12) the EC Lagrangian (2.8) plus an induced cosmological constant.[6] A similar reduction happens for the duality ansatz (3.2) as we have seen.

Since $R^{(\pm)}_{\alpha\beta} = \pm g^2/(g^2\theta_{\rm L}^{\star} \mp 1)\eta_{\alpha\beta}/\ell^2$, the weak coupling limit $g \to 0$ implies vanishing (chiral) curvature, cf. [83]. On the other hand, the weak coupling regime $\ell \to \ell_{\rm P} = 10^{-33}{\rm cm} \approx 0$ of macroscopic Einstein gravity implies for $g = 1$ a strong curvature scalar $R = 12(g/\ell)^2 = 48/\alpha' \to \infty$, i.e. the strong coupling regime of SKY gravity as part of the effective string or curvature-saturated Lagrangian, cf. Ref. [47]. Moreover, for the Taub–NUT metric this induces a rotation in the plane spanned by mass M and *dual mass* N (angular momentum), as felt by chiral fermions [60]. This duality of strong and weak coupling resembles that found by Montonen and Olive [76] in the contex of magnetic monopoles.

[6] In a rather ad hoc fashion, such a FR was applied already in Ref. [59] and later by Obukhov and Hehl [80] to Euler and Pontrjagin type terms. However, such deformations change the latter four-forms from being anymore boundary terms, thus preventing a topological interpretation in the spirit of S–duality.

5. Chromogravity and the Anti-De Sitter Model of Quark Confinement

The occurrence of an 'induced' cosmological constant (3.6) and its metrical Anti-de Sitter solution (AdS) has led to many speculation for its meaning in particle physics. It was already Einstein [21] who pointed that that ".. the scalar curvature plays the role of a negative pressure, ... whose gradient balances the electrodynamic force." Later on a similar idea was taken up by Salam et al. in his *strong gravity* model of quark confinement, cf. [56, 57, 77].

Independent of the physical interpretation, a 4D Anti-de Sitter background provides an *geometrodynamical mechanism* for quark confinement already on the semi-classical level. Then the *generally covariant* Klein–Gordon equation for a tortoise type radial coordinate ρ^* reduces to an effective Schrödinger equation with a Pöschl–Teller type effective potential $U_{\text{eff}} \sim 1 + \tan^2(\rho^*)$ familiar from oscillations of diatomic molecules. The energy spectrum is the same as that of a non-relativistic harmonic oscillator, see [61] for details. Since this potential 'wall' is infinitely rising, there exist equally spaced excited states but NO disintegration of the constituents can occur. Thus our AdS model is a fully relativistic model of an harmonic oscillator.

Today it is advocated to use the effective D=11 supergravity resulting from M-theory after compactification to $\text{AdS}_4 \otimes S^7$ space [96], as a calculational means ('analog computer') [78] for the strong coupling regime of *quark confinement* in QCD.

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Appendix A: Redefined Maxwell Fields

Let us consider as a guiding example Maxwell's or Yang–Mills theory in n dimensions, where $A := A^I_\mu dx^\mu \lambda_I$ is the Lie-algebra valued gauge potential one–form and $F := dA + A \wedge A$ the field strength. A rather general deformation or *field redefinition* (FR) of the gauge connection A which respects gauge-invariance and the form degree reads

$$A \quad \to \quad \widetilde{A} = A + \xi \rfloor {}^*H - {}^*DH + {}^*j \,, \tag{5.1}$$

where $H := -\partial G/\partial F$ is the excitation $(n-2)$–form of some *generating* n–form G. Quite generally, such a FR does not depend on a dimensionfull parameter and assures in the Abelian case that the new field strength $\widetilde{F} := d\widetilde{A} = F + d(\xi \rfloor {}^*H + {}^*DH - {}^*j)$ is derivable from a vector potential. In the case of a topological boundary term $G_\infty = \theta dC = -(\theta/2)Tr(F \wedge F)$ derived from a Chern–Simons term C, there will arise a θ induced FR with $H = \theta F$ and $DH = \theta DF \equiv 0$ due to the gauge-covariant Bianchi identity $DF \equiv 0$. Our construction generalizes the rather formal field redefinition $\overline{A} = A + \delta G/\delta j$ of Dixon [15] for non–Abelian gauge theories, where $j := \delta G/\delta A \cong -DH$ is a gauge current $(n-1)$–form. 'On shell', only the term depending on the vector field ξ will remain, due to $DH \cong j$. A related one-form $\hat{A} := {}^*j - {}^*dH^{\pm} = A + {}^*(A \wedge H^{\pm})$ has been recently used by Ganor and Sonnenschein [24] in order to replace A by $\hat{A}$ in a classical duality of gauge theories, cf. Eq.(11) of Ref. [75]. Also Born–Infeld type Lagrangians may be generated, cf. [28]. For Maxwell's theory the corresponding *canonical transformation* was given by Lozano [53] within a Hamiltonian formulation, whereas in the non-Abelian case the theory turns out to be of the Freedman–Townsend type [22] where the new two-forms are not derivable from a covector potential, or one-form. In our case the generating functional reads $G^{(\pm)} = H^{(\pm)}_{\alpha\beta} \wedge \widetilde{H}^{\alpha\beta}$.

If the vector field ξ is *normal* to a spacelike hypersurface, i.e. $\xi\rfloor dt = 1$, the normal part $A_\perp := \xi\rfloor A$ remains invariant under this deformation. The ansatz of Ashtekar and Rovelli [4] used for projecting out positive frequency fields in the Hamiltonian formulation can be viewed as such a FR for a special choice of G.

Experimentally accessible appears [16] to be the duality rotation of the Aharanov–Bohm/Casher topological phases in Maxwell's theory.

In effective string theory, the roles of the 'theta angle' in front of the topological boundary term $G_\infty = \theta dC$ and the gauge coupling constant g in the Yang–Mills Lagrangian

$$L_{\rm YM} = -\frac{1}{2g^2} Tr\,(F \wedge {}^*F) \tag{5.2}$$

are related to the vacuum expectation value (VEV) of the *axion* a and the dilaton field φ, respectively, via $\theta = 2\pi\langle 0|a|0\rangle$ and $g^2 = 4\pi\langle 0|e^{\varphi}|0\rangle = 32\pi G/\alpha' = 4\ell^2_{\rm Planck}\alpha'$, where $2\pi\alpha'$ is the string tension.

Appendix B: Geometry of a Riemann–Cartan Spacetime

Our geometrical arena consists of a four–dimensional manifold which is equipped with a local Riemannian metric (2.4) of Lorentz signature $(o_{\alpha\beta}) = \mathrm{diag}(-1, 1, 1, 1)$. For the representation of spinors in a curved spacetime, it is necessary to have the anholonomic formalism available on par. Therefore, we introduce an orthonormal local frame and coframe field of dimension [1/length] and [length], respectively

$$e_\alpha = e^i{}_\alpha\, \partial_i\,, \qquad \vartheta^\alpha = e_j{}^\alpha\, dx^j\,. \tag{5.3}$$

According to our conventions, $\alpha, \beta, ... = 0, 1, ..., 3$ are anholonomic frame indices, $i, j, k, ... = 0, 1, ..., 3$ are holononic or world indices, and $\wedge$ denotes the exterior product. The *coframe* field of basis one–forms are reciprocal to the frame e_α with respect to the *interior product* $\rfloor$, i.e., $e_\alpha\rfloor\vartheta^\beta = e^i{}_\alpha\, e_i{}^\beta = \delta^\beta_\alpha$.

In a Yang–Mills type gauge theory of gravity, the coframe ϑ^α of dimension [length] and the dimensionless connection one-form $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha} = \Gamma_i{}^{\alpha\beta}\, dx^i$ are regarded as gauge potentials of non–linearly represented *local translations* and *local Lorentz transformations*, respectively, cf. Ref. [61, 70, 52, 43]. The corresponding translational field strength is the *torsion* two–form

$$T^\alpha := D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta = \frac{1}{2}\, T_{ij}{}^\alpha \, dx^i \wedge dx^j , \tag{5.4}$$

of dimension [length] and the dimensionless Riemann–Cartan (RC) *curvature* two–form [13]

$$R^{\alpha\beta} := d\Gamma^{\alpha\beta} - \Gamma^{\alpha\gamma} \wedge \Gamma_\gamma{}^\beta = \frac{1}{2}\, R_{ij}{}^{\alpha\beta}\, dx^i \wedge dx^j . \tag{5.5}$$

These field strengths obey the *first* and *second Bianchi identities*

$$DT^\alpha \equiv R_\gamma{}^\alpha \wedge \vartheta^\gamma , \qquad \text{and} \qquad DR^{\alpha\beta} \equiv 0 \,. \tag{5.6}$$

The corresponding Lagrangians [33] are the Chern–Simons type boundary terms

$$dC_{\rm TT} = \frac{1}{2\ell^2}\left(T^\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta\right) =: V_{\rm NY}\,, \qquad dC_{\rm RR} = -\frac{1}{2} R_\alpha{}^\beta \wedge R_\beta{}^\alpha =: V_{\rm Pontrjagin}\,, \tag{5.7}$$

where ℓ is a fundamental length. Up to normalizations, they are also known as Nieh–Yan four-form [79] and Pontrjagin term, respectively. Both Chern–Simons terms arise as part of the CS term $\hat{C} = C_{\rm RR} - 2C_{\rm TT}$ of a gauge theory with $SL(5,R)$ as structure group, containing the de Sitter groups $SO(1,4)$ or $SO(2,3)$ as subgroups, see footnote 31 of Ref. [34] and Pagels [84] for a dynamical scheme.

The Riemannian content of our geometrical framework can be brought out by splitting the RC connection according to $\Gamma^{\alpha\beta} = \Gamma^{\{\}\,\alpha\beta} - K^{\alpha\beta}$ into the unique Levi–Civita connection $\Gamma^{\{\}\,\alpha\beta}$ of Riemannian geometry and into the *contortion*

$$K_{\alpha\beta} = -K^{\beta\alpha} = e_{[\alpha}\rfloor T_{\beta]} - \frac{1}{2}(e_\alpha \rfloor e_\beta \rfloor T_\gamma)\vartheta^\gamma \,. \tag{5.8}$$

It follows from (5.4) that the latter is related to torsion implicitly via $T^\alpha = K^\alpha{}_\beta \wedge \vartheta^\beta$. In turn, the RC curvature two–form (5.5) decomposes as follows

$$R^{\alpha\beta} = R^{\{\}\,\alpha\beta} + D^{\{\}}\, K^{\alpha\beta} + K^\alpha{}_\mu \wedge K^{\mu\beta} . \tag{5.9}$$

Appendix C: Dual Forms

On an n–dimensional manifold with metric index s, the Hodge dual of p–forms is almost involutive: ${}^{**}\alpha = (-1)^{p(n-p)+s}\alpha$. For spacetimes where $s = 1$ holds, it induces an *almost complex structure*, cf. [10]. In four dimensions, the *Hodge dual* applied to two–forms is *conformally invariant* [5]. Vice versa, an initially metric-free *involutive* star operation $^\#$ on arbitrary two–forms allows to *reconstruct* [17, 29] a metric h which is conformally related to g. Our *Hodge dual* * of exterior forms is defined such that the normalization

$${}^*(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta) = \eta^{\alpha\beta\gamma\delta} , \qquad \text{where} \quad \eta_{\alpha\beta\gamma\delta} := +\delta^{0123}_{\alpha\beta\gamma\delta} \quad \text{and} \quad D\eta_{\alpha\beta\gamma\delta} = 0 \tag{5.10}$$

holds.

From the volume four–form $\eta = \frac{1}{4!}\eta_{\alpha\beta\gamma\delta}\,\vartheta^\alpha\wedge\vartheta^\beta\wedge\vartheta^\gamma\wedge\vartheta^\delta$, the so-called η- or dual basis $\{\eta,\,\eta_\alpha,\,\eta_{\alpha\beta},\,\eta_{\alpha\beta\gamma}\,\eta_{\alpha\beta\gamma\delta}\}$ of exterior forms can now be generated by consecutive interior products: $\eta_\alpha := e_\alpha\rfloor\eta = {}^*\vartheta_\alpha$, $\eta_{\alpha\beta} := e_\beta\rfloor\eta_\alpha = \eta_{\alpha\beta\gamma}\,\vartheta^\gamma = e_\beta\rfloor e_\alpha\rfloor\eta = {}^*(\vartheta_\alpha\wedge\vartheta_\beta) = \frac{1}{2}\eta_{\alpha\beta\gamma\delta}\,\vartheta^\gamma\wedge\vartheta^\delta$, and $\eta_{\alpha\beta\gamma} := e_\gamma\rfloor\eta_{\alpha\beta} = {}^*(\vartheta_\alpha\wedge\vartheta_\beta\wedge\vartheta_\gamma)$. Anholonomic indices are lowered by $o_{\alpha\beta} = e^i{}_\alpha\,e^j{}_\beta\,g_{ij}$, where $(o_{\alpha\beta}) = \mathrm{diag}(-1,\,1,\,1,\,1)$ denotes the signature of spacetime.

The *Lie dual* of Lorentz algebra–valued forms such as contortion and curvature is defined by

$$K_\alpha^{(\star)} := \frac{1}{2}\eta_{\alpha\beta\gamma}\wedge K^{\beta\gamma}\,, \qquad R_{\alpha\beta}^{(\star)} := \frac{1}{2}\eta_{\alpha\beta\gamma\delta}R^{\gamma\delta}\,. \tag{5.11}$$

and satisfies $DR_{\alpha\beta}^{(\star)} \equiv 0$.

We will also employ the self– or anti-selfdual torsion and curvature two forms

$$T_\alpha^{\pm} := \frac{1}{2}\left(T_\alpha \pm {}^*T_\alpha\right)\,, \qquad R_{\alpha\beta}^{\pm} := \frac{1}{2}\left(R_{\alpha\beta} \pm {}^*R_{\alpha\beta}\right)\,, \qquad R_{\alpha\beta}^{(\pm)} := \frac{1}{2}\left(R_{\alpha\beta} \pm R_{\alpha\beta}^{(\star)}\right) \tag{5.12}$$

in terms of the Hodge or Lie dual, respectively. In view of this, the teleparallel boundary term and the topological Euler terms can be written as

$$dC_{\mathrm{TT}^*} := \frac{1}{2\ell^2}d(\vartheta^\alpha\wedge{}^*T_\alpha) = \frac{1}{2\ell^2}\left(T^\alpha\wedge{}^*T_\alpha - D\,{}^*T_\alpha\right)\,, \qquad dC_{\mathrm{RR}^\star} := -\frac{1}{2}\,R^{\alpha\beta}\wedge R_{\beta\alpha}^{(\star)} = V_{\mathrm{Euler}}\,. \tag{5.13}$$

From (5.9) results the geometric identity [34]

$$\begin{aligned} R^{\{\}\alpha\beta}\wedge\eta_{\alpha\beta} &\equiv R^{\alpha\beta}\wedge\eta_{\alpha\beta} - K^{\alpha\mu}\wedge K_\mu{}^\beta\wedge\eta_{\alpha\beta} + K^{\alpha\beta}\wedge T^\gamma\wedge\eta_{\alpha\beta\gamma} + d(K^{\alpha\beta}\wedge\eta_{\alpha\beta}) \\ &= R^{\alpha\beta}\wedge\eta_{\alpha\beta} + T^\alpha\wedge{}^*\Big(-{}^{(1)}T_\alpha + 2\,{}^{(2)}T_\alpha + \frac{1}{2}\,{}^{(3)}T_\alpha\Big) + 4\ell^2\,dC_{\mathrm{TT}^*} \end{aligned} \tag{5.14}$$

which relates the Hilbert–Einstein Lagrangian to the proper *teleparallelism* model, with the Lagrangian constraint $R^{\alpha\beta} = 0$. These topological terms have been proposed, for instance in Ref. [63], in the combination

$$V_\theta := \theta_{\mathrm{T}}\,dC_{\mathrm{TT}} + \theta_{\mathrm{T}}^*\,dC_{\mathrm{TT}^*} + \theta_{\mathrm{L}}\,dC_{\mathrm{RR}} + \theta_{\mathrm{L}}^\star dC_{\mathrm{RR}^\star} \tag{5.15}$$

as *generating function* for obtaining more general Ashtekar type variables in the Hamiltonian formulation for $\theta_{\mathrm{L}}^\star = 0$. With exception of the Euler terms, these θ terms *violate parity*, but for purely *imaginary* θ parameters they would conserved the combined charge conjugation C and parity transformation P, i.e. CP, cf. [71]. Some of these terms arise also in the chiral anomaly [68, 64].

Appendix D: Framework of Gravitational Gauge Dynamics

The total action of interacting matter and gravitational gauge fields

$$W = \int\Big[L(\vartheta^\alpha,\Psi,D\Psi) - V(\vartheta^\alpha,T^\alpha,R^{\alpha\beta})\Big] \tag{5.16}$$

is assumed to be a functional of suitable matter fields Ψ and of the geometrical variables ϑ^α and $\Gamma^{\alpha\beta}$. Besides the Euler-Lagrange equation $\delta L/\delta\Psi = 0$ for matter, their *independent* variations yield the following two, general *nonlinear* field equations:

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \tag{5.17}$$

$$DH_{\alpha\beta} + \vartheta_{[\alpha} \wedge H_{\beta]} = \tau_{\alpha\beta}. \tag{5.18}$$

Here the *gauge field momenta* are defined by the $(n-2)$–forms:

$$H_\alpha := -\frac{\partial V}{\partial T^\alpha}, \qquad \text{and} \qquad H_{\alpha\beta} := -\frac{\partial V}{\partial R^{\alpha\beta}}. \tag{5.19}$$

Note that in $n = 4$ dimensions H_α has dimension [length]. In addition to the material currents of energy–momentum $\Sigma_\alpha := \partial L/\partial\vartheta^\alpha$ and dynamical spin $\tau_{\alpha\beta} := \partial L/\partial\Gamma^{\alpha\beta}$, there occur the three–forms of *energy–momentum* $E_\alpha := \partial V/\partial\vartheta^\alpha = e_\alpha\rfloor V + (e_\alpha\rfloor T^\beta) \wedge H_\beta + (e_\alpha\rfloor R^{\beta\gamma}) \wedge H_{\beta\gamma}$ and the *translational spin current* $E_{\alpha\beta} = -\vartheta_{[\alpha} \wedge H_{\beta]}$ of the gravitational gauge fields themselves [35, 32]. This is due to the universality of gravitational interactions.

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