

A Review of Leading Quantum Gravitational Corrections to Newtonian Gravity

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Abstract: In this review we present the theoretical background for treating General Relativity as an effective field theory and focus on the concrete results of such a treatment. As a result we present the calculations of the low-energy leading gravitational corrections to the Newtonian potential between two sources.

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1. Introduction

The fundamental equation of the non-relativistic theory of gravity is the Newtonian gravitational law, which predicts the potential energy of the gravitational attraction between two bodies as:

$$V(r) = -G \frac{m_1 m_2}{r} \quad (1)$$

Here $V(r)$ is a measure for the potential energy, m_1 and m_2 are the masses of the two particles, r is the distance between the masses and G is the universal gravitational constant.

In contrast the theory of General Relativity [1] provides a framework for extending Newton's theory to objects with relativistic velocities. In general relativity one solves the basic field equation:

$$R_{\mu\nu}(g_{\mu\nu}) - \frac{1}{2}R(g_{\mu\nu})g_{\mu\nu} = 16\pi GT_{\mu\nu} - \Lambda g_{\mu\nu} \quad (2)$$

where $g_{\mu\nu}$ is the gravitational metric, $R_{\beta\mu\nu}^{\alpha}$ is the tensor for the curvature of space-time¹ and $T_{\mu\nu}$ is the total energy-momentum tensor. The cosmological constant Λ may be needed on cosmological scales, and is today believed to have a non-zero expectation value in the Universe. When we solve the Einstein equation we find the metric which is a local object that depends on the geometry of space-time. In this way a solution of the gravitational problem is found. Einstein's description holds in the fully relativistic regime, and its low-energy and non-relativistic predictions match the expectations of Newtonian mechanics.

A longstanding puzzle in Modern Physics is how to wed General Relativity with the quantum theory. It is not at all obvious how this might be achieved since General Relativity and quantum mechanics seem to be based on completely different perceptions of physics – nevertheless this question is one of the most pressing questions of modern theoretical physics and has been the subject of many studies, *e.g.*, see refs. [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

All sorts of interpretational complications arise when trying to quantize General Relativity. A possible starting point for such a theory appears to be to interpret General Relativity as a quantum field theory, to let the metric be the basic gravitational field, and to quantize the Einstein-Hilbert action:

$$\mathcal{S}_{EH} = \int d^4x \sqrt{-g} \frac{R}{16\pi G} \quad (3)$$

where $g = \det(g_{\mu\nu})$ and R is the scalar curvature. However the above action is not self contained under renormalization since loop diagrams will generate new terms not present in the original action refs. [10, 11, 25, 26]. This is the renowned renormalization problem that hinders the quantization of general relativity.

One of the physically interesting problem is the calculation of the leading order quantum corrections to the Newtonian potential which has been in the focus of many studies in different schemes, using Feynman diagrams for the loops in the graviton propagator [12, 13, 14, 15, 16, 17, 10], renormalizable R^2 gravity [18, 19, 20] and Semiclassical Gravity [21, 22, 23, 24].

After introducing an effective field theory for processes with a typical energy less the Planck mass, i.e. with $|q^2| \ll M_P^2 \simeq 10^{38} \text{ GeV}^2$, by Weinberg [27], the effective theory for gravity can be modeled in a manner analogous with that of Chiral Perturbation Theory [28] for QCD. This way to look at General Relativity was proposed by Shiekh [29] and Donoghue [30], and they have shown that reliable quantum predictions at the low energies can be made.

In spite of fact that unmodified General Relativity is not renormalizable, be it pure General Relativity or General Relativity coupled to bosonic or fermionic matter, see *e.g.* [10, 11, 31, 32], using the framework of an effective field theory, these theories do become order by order renormalizable in the low energy limit. When General Relativity is treated as an effective theory, renormalizability simply fails to be an issue. The ultraviolet

¹ $(R_{\mu\nu} = R^{\beta}_{\mu\nu\beta}), (R \equiv g^{\mu\nu} R_{\mu\nu})$

divergences arising e.g. at the 1-loop level are dealt with by renormalizing the parameters of higher derivative terms in the action.

When approaching general relativity in this manner, it is convenient to use the background field method [2, 33]. Divergent terms are absorbed away into phenomenological constants which characterize the effective action of the theory. The price paid is the introduction of a set of never-ending higher order derivative couplings into the theory, unless using the approach of Shiekh [29]. The effective action contains all terms consistent with the underlying symmetries of the theory. Perturbatively only a finite number of terms in the action are required for each loop order.

In pioneering papers [30] Donoghue first has shown how to derive the leading quantum and classical relativistic corrections to the Newtonian potential of two masses. This calculation has since been the focus of a number of publications [34, 35, 36, 37, 38, 39], and this work continues, most recently in the paper [40].

Unfortunately, due to difficulty of the calculation and its myriad of tensor indices there has been some disagreement among the results of various authors. The classical component of the corrections were found long ago by Einstein, Infeld and Hoffmann [41], and by Eddington and Clark [42]. Later this result was reproduced by Iwasaki [9] by means of Feynman diagrams and has been discussed in the papers [43, 44, 45], and here there is general agreement although there exists an unavoidable ambiguity in defining the potential.

An interesting calculation has been made involving quantum gravitational corrections to the Schwarzschild and Kerr metrics of scalars and fermions [46, 47] where it is shown in detail how the higher order gravitational contributions to these metrics emerge from loop calculations. In the papers [48] and [49] have been calculated the leading post-Newtonian and quantum corrections to the non-relativistic scattering amplitude of charged scalars and spin- $\frac{1}{2}$ fermions in the combined theory of general relativity and QED. For the recent reviews of general relativity as an effective field theory, see refs. [50, 51]

Our notations and conventions on the metric tensor, the gauge-fixed gravitational action, etc. are the same as in [36], namely ($\hbar = c = 1$) as well as the Minkowski metric convention (+1, -1, -1, -1).

2. The Quantization of General Relativity

The Einstein action for General Relativity has the form:

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{2R}{\kappa^2} + \mathcal{L}_{\text{matter}} \right] \quad (4)$$

where $\kappa^2 = 32\pi G$ is defined as the gravitational coupling, and the curvature tensor is defined as:

$$R^\mu_{\nu\alpha\beta} \equiv \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\sigma\alpha} \Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta} \Gamma^\sigma_{\nu\alpha} \quad (5)$$

and

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\sigma} (\partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta}) \quad (6)$$

The term $\sqrt{-g}\mathcal{L}_{\text{matter}}$ is a covariant expression for the inclusion of matter into the theory. We can include any type of matter. As a classical theory the above Lagrangian defines the theory of general relativity.

Massive spinless matter fields interact with the gravitational field as described by the action

$$\mathcal{S}_{\text{matter}} = \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right] \quad (7)$$

Any effective field theory can be seen as an expansion in energies of the light fields of the theory below a certain scale. Above the scale transition energy there will be additional heavy fields that will manifest themselves. Below the transition the heavy degrees of freedom will be integrated out and will hence not contribute to the physics. Any effective field theory is built up from terms with higher and higher numbers of derivative couplings on the light fields and obeying the gauge symmetries of the basic theory. This gives us a precise description of how to construct effective Lagrangians from the gauge invariants of the theory. We expand the effective Lagrangian in the invariants ordered in magnitude of their derivative contributions.

An effective treatment of pure General Relativity results in the following Lagrangian:

$$\mathcal{L}_{\text{grav}} = \sqrt{-g} \left[\frac{2R}{\kappa^2} + c_1 R^2 + c_2 R^{\mu\nu} R_{\mu\nu} + \dots \right] \quad (8)$$

where the ellipses denote that the effective action is in fact an infinite series—at each new loop order additional higher derivative terms must be taken into account. This Lagrangian includes all possible higher derivative couplings, and every coupling constant in the Lagrangian is considered to be determined empirically unless set to zero to achieve causality [29]. Similarly one must include higher derivative contributions to the matter Lagrangian in order to treat this piece of the Lagrangian as an effective field theory [30].

Computing the leading low-energy quantum corrections of an effective field theory, a useful distinction is between non-analytical and analytical contributions from the diagrams. Non-analytical contributions are generated by the propagation of two or more massless particles in the Feynman diagrams. Such non-analytical effects are long-ranged and, in the low energy limit of the effective field theory, they dominate over the analytical contributions which arise from the propagation of massive particles. The difference between massive and massless particle modes originates from the impossibility of expanding a massless propagator $\sim 1/q^2$ while:

$$\frac{1}{q^2 - m^2} = -\frac{1}{m^2} \left(1 + \frac{q^2}{m^2} + \dots \right) \quad (9)$$

No $1/q^2$ terms are generated in the above expansion of the massive propagator, thus such terms all arise from the propagation of massless modes. The analytical contributions from the diagrams are local effects and thus expandable in power series.

Non-analytical effects are typically originating from terms which in the S-matrix go as, *e.g.*, $\sim \ln(-q^2)$ or $\sim 1/\sqrt{-q^2}$, while the generic example of an analytical contribution

is a power series in momentum q . Our interest is only in the non-local effects, thus we will only consider the non-analytical contributions of the diagrams.

The procedure of the background field quantization is as follows. The quantum fluctuations of the gravitational field are expanded about a smooth background metric $\bar{g}_{\mu\nu}$ [10, 11], i.e. flat space-time $\bar{g}_{\mu\nu} \equiv \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, and the metric $g_{\mu\nu}$ is the sum of this background part and a quantum contribution $\kappa h_{\mu\nu}$:

$$g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \quad (10)$$

From this equation we get the expansions for the upper metric field $g^{\mu\nu}$, and for $\sqrt{-g}$:

$$\begin{aligned} g^{\mu\nu} &= \bar{g}^{\mu\nu} - \kappa h^{\mu\nu} + \dots \\ \sqrt{-g} &= \sqrt{-\bar{g}} \left[1 + \frac{1}{2} \kappa h + \dots \right] \end{aligned} \quad (11)$$

where $h^{\mu\nu} \equiv \bar{g}^{\mu\alpha} \bar{g}^{\nu\beta} h_{\alpha\beta}$ and $h \equiv \bar{g}^{\mu\nu} h_{\mu\nu}$.

The corresponding curvatures are given by

$$\begin{aligned} \bar{R}_{\mu\nu} &= \frac{\kappa}{2} [\partial_\mu \partial_\nu h + \partial_\lambda \partial^\lambda h_{\mu\nu} - \partial_\mu \partial_\lambda h^\lambda{}_\nu - \partial_\nu \partial_\lambda h^\lambda{}_\mu] \\ \bar{R} &= \bar{g}^{\mu\nu} \bar{R}_{\mu\nu} = \kappa [\square h - \partial_\mu \partial_\nu h^{\mu\nu}] \end{aligned} \quad (12)$$

In order to quantize the field $h_{\mu\nu}$ one needs to fix the gauge. In the harmonic (or deDonder) gauge [10] $-g^{\mu\nu} \Gamma_{\mu\nu}^\lambda = 0$ —which requires, to first order in the field expansion,

$$\partial^\beta h_{\alpha\beta} - \frac{1}{2} \partial_\alpha h = 0 \quad (13)$$

In the quantization, the Lagrangians are expanded in the gravitational fields, separated in quantum and background parts, and the vertex factors as well as the propagator are derived from the expanded action.

The expansion of the Einstein action takes the form [10, 11]:

$$\mathcal{S}_{grav} = \int d^4x \sqrt{-\bar{g}} \left[\frac{2\bar{R}}{\kappa^2} + \mathcal{L}_g^{(1)} + \mathcal{L}_g^{(2)} + \dots \right] \quad (14)$$

where the subscripts count the number of powers of κ and

$$\begin{aligned} \mathcal{L}_g^{(1)} &= \frac{h_{\mu\nu}}{\kappa} [\bar{g}^{\mu\nu} \bar{R} - 2\bar{R}^{\mu\nu}] \\ \mathcal{L}_g^{(2)} &= \frac{1}{2} D_\alpha h_{\mu\nu} D^\alpha h^{\mu\nu} - \frac{1}{2} D_\alpha h D^\alpha h + D_\alpha h D_\beta h^{\alpha\beta} - D_\alpha h_{\mu\beta} D^\beta h^{\mu\alpha} \\ &\quad + \bar{R} \left(\frac{1}{4} h^2 - \frac{1}{2} h_{\mu\nu} h^{\mu\nu} \right) + (2h_\mu^\lambda h_{\nu\lambda} - h h_{\mu\nu}) \bar{R}^{\mu\nu} \end{aligned} \quad (15)$$

where D_α denotes the covariant derivative with respect to the background metric. A similar expansion of the matter action yields [30]:

$$\mathcal{S}_{matter} = \int d^4x \sqrt{-\bar{g}} [\mathcal{L}_m^0 + \mathcal{L}_m^{(1)} + \mathcal{L}_m^{(2)} + \dots] \quad (16)$$

with

$$\begin{aligned} \mathcal{L}_m^{(0)} &= \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) \\ \mathcal{L}_m^{(1)} &= -\frac{\kappa}{2} h_{\mu\nu} T^{\mu\nu} \\ T_{\mu\nu} &\equiv \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \bar{g}_{\mu\nu} (\partial_\lambda \phi \partial^\lambda \phi - m^2 \phi^2) \\ \mathcal{L}_m^{(2)} &= \kappa^2 \left(\frac{1}{2} h^{\mu\nu} h_\lambda^\nu - \frac{1}{4} h h^{\mu\nu} \right) \partial_\mu \phi \partial_\nu \phi \\ &\quad - \frac{\kappa^2}{8} \left(h^{\lambda\sigma} h_{\lambda\sigma} - \frac{1}{2} h h \right) [\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2] \end{aligned} \quad (17)$$

The background metric $\bar{R}^{\mu\nu}$ should satisfy Einstein's equation

$$\bar{R}^{\mu\nu} - \frac{1}{2} \bar{g}^{\mu\nu} \bar{R} = \frac{\kappa^2}{4} T^{\mu\nu} \quad (18)$$

and the linear terms in $h_{\mu\nu}, \mathcal{L}_g^{(1)} + \mathcal{L}_m^{(1)}$, is vanishing.

For the calculation of the quantum gravitational corrections at one loop, we need to consider the following actions:

$$\begin{aligned} \mathcal{S}_0 &= \int d^4x \sqrt{-g} \left[\frac{2\bar{R}}{\kappa^2} + \mathcal{L}_m^{(0)} \right] \\ \mathcal{S}_2 &= \int d^4x \sqrt{-g} [\mathcal{L}_g^{(2)} + \mathcal{L}_m^{(2)} + \mathcal{L}_{gauge} + \mathcal{L}_{ghost}] \end{aligned} \quad (19)$$

with the gauge fixing Lagrangian [10]

$$\mathcal{L}_{gauge} = \left(D^\nu h_{\mu\nu} - \frac{1}{2} D_\mu h \right) \left(D_\lambda h^{\mu\lambda} - \frac{1}{2} D^\mu h \right) \quad (20)$$

and the ghost Lagrangian

$$\mathcal{L}_{ghost} = \eta^{*\mu} (D_\lambda D^\lambda \eta_\mu - \bar{R}_{\mu\nu} \eta^\nu) \quad (21)$$

for the Faddeev-Popov field η_μ .

3. The Feynman Rules

From the Lagrangians (19)-(21) we can derive the list of Feynman rules [38].

- **Scalar propagator**

The massive scalar propagator is:

where

$$\begin{aligned}
\tau_{\alpha\beta\gamma\delta}^{\mu\nu}(k, q) = & -\frac{i\kappa}{2} \times \left(\mathcal{P}_{\alpha\beta\gamma\delta} \left[k^\mu k^\nu + (k - q)^\mu (k - q)^\nu + q^\mu q^\nu - \frac{3}{2} \eta^{\mu\nu} q^2 \right] \right. \\
& + 2q_\lambda q_\sigma \left[I_{\alpha\beta}^{\sigma\lambda} I_{\gamma\delta}^{\mu\nu} + I_{\gamma\delta}^{\sigma\lambda} I_{\alpha\beta}^{\mu\nu} - I_{\alpha\beta}^{\mu\sigma} I_{\gamma\delta}^{\nu\lambda} - I_{\gamma\delta}^{\mu\sigma} I_{\alpha\beta}^{\nu\lambda} \right] \\
& + \left[q_\lambda q^\mu \left(\eta_{\alpha\beta} I_{\gamma\delta}^{\nu\lambda} + \eta_{\gamma\delta} I_{\alpha\beta}^{\nu\lambda} \right) + q_\lambda q^\nu \left(\eta_{\alpha\beta} I_{\gamma\delta}^{\mu\lambda} + \eta_{\gamma\delta} I_{\alpha\beta}^{\mu\lambda} \right) \right. \\
& \left. - q^2 \left(\eta_{\alpha\beta} I_{\gamma\delta}^{\mu\nu} - \eta_{\gamma\delta} I_{\alpha\beta}^{\mu\nu} \right) - \eta^{\mu\nu} q_\sigma q_\lambda \left(\eta_{\alpha\beta} I_{\gamma\delta}^{\sigma\lambda} + \eta_{\gamma\delta} I_{\alpha\beta}^{\sigma\lambda} \right) \right] \\
& + \left[2q_\lambda \left(I_{\alpha\beta}^{\lambda\sigma} I_{\gamma\delta\sigma}^{\nu} (k - q)^\mu + I_{\alpha\beta}^{\lambda\sigma} I_{\gamma\delta\sigma}^{\mu} (k - q)^\nu - I_{\gamma\delta}^{\lambda\sigma} I_{\alpha\beta\sigma}^{\nu} k^\mu - I_{\gamma\delta}^{\lambda\sigma} I_{\alpha\beta\sigma}^{\mu} k^\nu \right) \right. \\
& \left. + q^2 \left(I_{\alpha\beta\sigma}^{\mu} I_{\gamma\delta}^{\nu\sigma} + I_{\alpha\beta}^{\nu\sigma} I_{\gamma\delta\sigma}^{\mu} \right) + \eta^{\mu\nu} q_\sigma q_\lambda \left(I_{\alpha\beta}^{\lambda\rho} I_{\gamma\delta\rho}^{\sigma} + I_{\gamma\delta}^{\lambda\rho} I_{\alpha\beta\rho}^{\sigma} \right) \right] \\
& + \left\{ (k^2 + (k - q)^2) \left[I_{\alpha\beta}^{\mu\sigma} I_{\gamma\delta\sigma}^{\nu} + I_{\gamma\delta}^{\mu\sigma} I_{\alpha\beta\sigma}^{\nu} - \frac{1}{2} \eta^{\mu\nu} \mathcal{P}_{\alpha\beta\gamma\delta} \right] \right. \\
& \left. - \left(I_{\gamma\delta}^{\mu\nu} \eta_{\alpha\beta} k^2 + I_{\alpha\beta}^{\mu\nu} \eta_{\gamma\delta} (k - q)^2 \right) \right\}
\end{aligned} \tag{23}$$

4. Scattering Amplitude and Potential

The general form for any diagram contributing to the scattering amplitude of gravitational interactions of two masses is:

$$\mathcal{M} \sim \left(A + Bq^2 + \dots + C_0 \kappa^4 \frac{1}{q^2} + C_1 \kappa^4 \ln(-q^2) + C_2 \kappa^4 \frac{m}{\sqrt{-q^2}} + \dots \right) \tag{24}$$

where A, B, \dots correspond to the local analytical interactions which are of no interest to us (these terms will only dominate in the high energy regime of the effective theory) and C_0, C_1, C_2, \dots correspond to the non-local, non-analytical interactions.

The C_1 and C_2 terms will yield the leading quantum gravitational and relativistic post-Newtonian corrections to the Newtonian potential. The space parts of the non-analytical terms Fourier transform as:

$$\begin{aligned}
\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \frac{1}{|\mathbf{q}|^2} &= \frac{1}{4\pi r} \\
\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \frac{1}{|\mathbf{q}|} &= \frac{1}{2\pi^2 r^2} \\
\int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \ln(\mathbf{q}^2) &= \frac{-1}{2\pi r^3}
\end{aligned} \tag{25}$$

so clearly these terms will contribute to the corrections.

The importance of these transforms, is that they are from non-analytic terms in momentum space and so cannot be renormalized into the original Lagrangian, and as

such one might anticipate that they are of finite magnitude. Because of this, the problem of renormalizing quantum gravity is put off.

In the quantization of General Relativity the definition of a potential is certainly not obvious. One can choose between several definitions of the potential depending on, *e.g.*, the physical situation, how to define the energy of the fields, the diagrams included etc. The choice of potential, which includes all 1-loop diagrams [35, 52], is the simplest, gauge invariant definition of the potential.

The calculation of the non-relativistic potential using the full amplitude is as follows. First, to relate the expectation value for the S matrix to the Fourier transform of the potential $\tilde{V}(\mathbf{q})$ in the non-relativistic limit:

$$\langle k_1, k_3 | S | k_2, k_4 \rangle = -i\tilde{V}(\mathbf{q})(2\pi)\delta(E_i - E_f) \quad (26)$$

where k_1, k_3 and k_2, k_4 are the incoming and outgoing momentum respectively, $q \equiv k_2 - k_1 = k_3 - k_4$, and $E_i - E_f$ is the energy difference between the incoming and outgoing states. The invariant matrix element $i\mathcal{M}$ resulting from the diagrams is:

$$\langle k_1, k_3 | S | k_2, k_4 \rangle = (2\pi)^4 \delta^{(4)}(k_2 + k_4 - k_1 - k_3)(i\mathcal{M}) \quad (27)$$

In the non-relativistic limit ($q = (0, \mathbf{q})$) we have:

$$\tilde{V}(\mathbf{q}) = -\frac{1}{2m_1} \frac{1}{2m_2} \mathcal{M} \quad (28)$$

so that

$$V(\mathbf{x}) = -\frac{1}{2m_1} \frac{1}{2m_2} \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \mathcal{M} \quad (29)$$

This is how we define the non-relativistic potential $V(\mathbf{q})$ generated by the considered non-analytic parts. In the above equation \mathcal{M} is the non-analytical part of the amplitude of the scattering process in non-relativistic limit to a given loop order [35].

5. The Contributions of Feynman Diagrams

In general, the Feynman rules are large and the tensor algebra immense. Much of the calculational simplicity should be restored by submitting this part of the complexity to the computer. However, the intermediate results can be so extensive that even a super-computer can choke without help. For example, imagine one had the contraction of three tensors: $\alpha^{\mu\nu}\beta^{\rho\sigma}\gamma_{\mu\nu\rho\sigma}$ each of which consists of many terms. Then the computer, in trying to contract out the indices, tends to expand out the entire expression, which can easily lead to thousands of terms in the intermediate expression, and so overpower the computers memory. The resolution lies in asking the computer to initially expand out only α for example: $(\alpha_1^{\mu\nu} + \alpha_2^{\mu\nu} + \dots)\beta^{\rho\sigma}\gamma_{\mu\nu\rho\sigma}$. In this way the computer is presented with several terms that can each be contracted separately. This seemingly innocuous move can make all the difference between the machine being able to perform the calculation or not. It is fine details like this that in practice can occupy much of the investigators time.

The best way to perform such kind of tensor algebra on a computer is use the Ricci package [53] under the Mathematica [54] program.

5.1 Vacuum Polarization

't Hooft and Veltman [10] were the first to calculate the vacuum polarization diagram in gravity. For the contribution of the graviton plus ghost vacuum polarization Feynman diagrams²:



we have:

$$\begin{aligned} \Pi_{\alpha\beta\gamma\delta} = & -\frac{\kappa^2}{16\pi^2} L \left[\frac{21}{120} q^4 I_{\alpha\beta\gamma\delta} + \frac{23}{120} q^4 \eta_{\alpha\beta} \eta_{\gamma\delta} - \frac{23}{120} q^2 (\eta_{\alpha\beta} q_\gamma q_\delta + \eta_{\gamma\delta} q_\alpha q_\beta) \right. \\ & \left. - \frac{21}{240} q^2 (q_\alpha q_\delta \eta_{\beta\gamma} + q_\beta q_\delta \eta_{\alpha\gamma} + q_\alpha q_\gamma \eta_{\beta\delta} + q_\beta q_\gamma \eta_{\alpha\delta}) + \frac{11}{30} q_\alpha q_\beta q_\gamma q_\delta \right] \end{aligned} \quad (30)$$

where $L \equiv \log(-q^2) = \ln(\mathbf{q}^2)$

The result after contracting the various indices is [30, 36, 38, 37]:

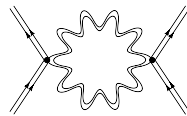
$$\mathcal{M}_{vac}(\mathbf{q}) = \frac{43}{15} G^2 m_1 m_2 L \quad (31)$$

The Fourier transform gives the following contribution to the scattering potential

$$V_{vac}(r) = -\frac{43}{30\pi} G^2 \frac{m_1 m_2}{r^3} \quad (32)$$

5.2 Double-seagull Contribution

The calculation of the double-seagull loop diagram:



is quite straightforward. The resulting amplitude is

$$\mathcal{M}_{seag}(\mathbf{q}) = 44G^2 m_1 m_2 L \quad (33)$$

whose Fourier transform yields the double-seagull contribution to the potential [37, 38]:

$$V_{seag}(r) = -\frac{22}{\pi} G^2 \frac{m_1 m_2}{r^3} \quad (34)$$

² The Feynman graphs have been plotted with **JaxoDraw** [55]

5.3 The Triangle Diagrams

The calculation of the triangle loop diagram:



yields no real complications:

$$\begin{aligned} \mathcal{M}_{tri}^a(\mathbf{q}) &= -8G^2 m_1 m_2 \left(\frac{7}{2}L + \frac{\pi^2 m_1}{|\mathbf{q}|} \right) \\ \mathcal{M}_{tri}^b(\mathbf{q}) &= -8G^2 m_1 m_2 \left(\frac{7}{2}L + \frac{\pi^2 m_2}{|\mathbf{q}|} \right) \end{aligned} \quad (35)$$

and the Fourier transformed result is [30, 36, 37, 38]:

$$V_{tri}(r) = -4G^2 \frac{m_1 m_2 (m_1 + m_2)}{r^2} + \frac{28}{\pi} G^2 \frac{m_1 m_2}{r^3} \quad (36)$$

5.4 Vertex Corrections

Two classes of diagrams go into the set of vertex corrections. There are two diagrams with a massless graviton in the loop:



The calculation of these diagrams is sufficiently simple and results in:

$$\mathcal{M}_{vert}^a(\mathbf{q}) = -\frac{52}{3} G^2 m_1 m_2 L \quad (37)$$

Much more tedious is the calculation of the vertex diagrams with massive particle in the loop:



The result is:

$$\mathcal{M}_{vert}^a(\mathbf{q}) = 2G^2 m_1 m_2 \left(\frac{5}{3}L + \frac{\pi^2 (m_1 + m_2)}{|\mathbf{q}|} \right) \quad (38)$$

The vertex diagrams are among the most complicated to calculate. The first results for these diagrams date back to the original calculation of Donoghue [30] — but because of an algebraic error in the calculation, the original result was in error and despite various checks of the calculation [36, 37] the correct result has not been given until [38].

The Fourier transform yields the following result for the vertex modification of the scattering potential [38, 39]:

$$V_{vert}^a(r) = \frac{26}{3\pi} G^2 \frac{m_1 m_2}{r^3} \quad (39)$$

and

$$V_{vert}^b(r) = G^2 \frac{m_1 m_2 (m_1 + m_2)}{r^2} - \frac{5}{3\pi} G^2 \frac{m_1 m_2}{r^3} \quad (40)$$

5.5 The Box Diagrams

The contribution of the box and crossed box diagrams:



to the scattering amplitude in the non-relativistic limit is:

$$\mathcal{M}_{box}(\mathbf{q}) = \frac{94}{3} G^2 m_1 m_2 L \quad (41)$$

and to the potential [37, 38]:

$$V_{box}(r) = -\frac{47}{3\pi} G^2 \frac{m_1 m_2}{r^3} \quad (42)$$

6. The Gravitational Corrections

Adding up all one-loop gravitational corrections we have the final result for the non-relativistic Newtonian potential [38]:

$$V(r) = -G \frac{m_1 m_2}{r} \left[1 + 3 \frac{G(m_1 + m_2)}{c^2 r} + \frac{41}{10\pi} \frac{G\hbar}{c^3 r^2} \right] \quad (43)$$

In the above expressions we have restored the appropriate physical factors c and \hbar .

On the grounds of dimensional analysis alone one can anticipate this form of the lowest-order corrections to the Newtonian potential [30]. The relativistic classical corrections are proportional to ℓ_{cl}/r , where

$$\ell_{cl} = \frac{Gm}{c^2} \quad (44)$$

is the classical length for the mass m , and the quantum corrections (also relativistic) are proportional to ℓ_p^2/r^2 , where

$$\ell_p = \sqrt{\frac{G\hbar}{c^3}} \quad (45)$$

$\ell_p = 1.6 \times 10^{-35}$ m is the Planck length.

The classical and quantum pieces of (43) arise from the same loop diagrams, and the order of magnitude of the quantum corrections $G\hbar/c^3 r^2$ can be derived from the classical one using [56] the concept of “zitterbewegung”. In fact, in transition from classical to quantum corrections the classical distance r between two masses m_1 and m_2 must be modified by an uncertainty of the order the Compton wavelengths of each masses:

$$r \rightarrow r + \frac{\hbar}{m_1 c} + \frac{\hbar}{m_2 c} \quad (46)$$

and

$$\frac{1}{r} \rightarrow \frac{1}{r} - \frac{\hbar}{(m_1 + m_2)cr^2} + \dots, \quad (47)$$

so that the quantum corrections of can be understood as “zitterbewegung” effects applied to the classical distance r .

It should be noticed that the classical post-Newtonian term in the expression (43) corresponds to the lowest-order scattering potential and agrees with Eq. 2.5 of Iwasaki [9]. The correct result for the quantum corrections first published in [38] and later was confirmed in [39].

Conclusion and Outlook

The result (41) for the leading quantum corrections to the Newton law could be written in the form:

$$V(r) = -G \frac{m_1 m_2}{r} [1 + \delta_{QC}] \quad (48)$$

where

$$\delta_{QC} = \frac{41}{10\pi} \frac{\ell_p^2}{r^2} \quad (49)$$

There are also additional quantum corrections due to the contributions to the vacuum polarization by photons and massless neutrinos:



which were calculated by Radkowski [12], Capper, Duff, and Halpern [15], Capper and Duff [16], Duff and Liu [17]:

$$\delta_{QC}^{\gamma\nu} = \frac{4 + N_\nu}{15\pi} \frac{\ell_p^2}{r^2} \quad (50)$$

where N_ν is the number of massless two-component neutrinos.

The value of the both quantum corrections are controlled by the Planck length ℓ_p , the corrections vanish at large values of r and it is accompanied by a very small coefficient, so even for astronomical purposes these corrections are irrelevant and unlikely to be

measured in the foreseeable future. Nevertheless, such predictions would need to be replicated by any candidate theory of high energy quantum gravity.

Only at $r_0 \simeq \ell_p$ the quantum corrections become large. But in this regime the effective field theory approach breaks down.

However, from a cosmological view point there is a cumulative effect of gravity and, given a fixed density of energy, the integration of this effect over large volumes could give an observable signal [57]. The authors of ref. [57] have found that during inflation, the quantum corrections are significant, leading to deviations from the standard inflationary expansion.

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