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The Restricted Three Body Problem with Quadratic Drag

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Abstract: When an asteroid, space-craft or another small object in the solar system is in the vicinity of a planet it is subjected to the gravitational forces of the Sun, the planet, the drag forces due to the solar wind and (possibly) the planet upper atmosphere. To determine the object trajectory we consider this problem within the context of the restricted three body problem in three dimensions with quadratic drag. In this setting we linearize the equations of motion of the object and cast them in a coordinate system with respect to the secondary (planet) which is assumed to be in a general Keplerian orbit around the primary (Sun). We then reduce them, to a simple system of three second order linear differential equations. These equations can be considered to be a generalization of Hill's equations to general Keplerian orbits (of the secondary) with the addition of quadratic drag force acting on the third object in the system. We derive also "approximate conservation laws" in three dimensions which represent a generalization of Jacobi's integral in two dimensions and consider some special cases.

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1. Introduction

The present paper is being motivated by the emerging realization that many "small" objects in the solar system whose size vary from few millimeters to few kilometers have orbits around the Sun which criss-cross the orbit of the various planets in the system. Although the smaller objects in this group might not pose any danger it is important to compute accurately their trajectories and the prospects of their capture when they are close enough to a planet. We note that it is not feasible (nor important from this point of view) to calculate these trajectories when they are far away from the planet as they

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are subjected then to many "small" perturbations.

From another point of view the trajectory of a space craft sent from Earth to one of the planets should be the subject of similar considerations as it nears it destination.

It is obvious that this problem should be treated within the context of the restricted three body problem (since the object mass is small). However in addition to the gravitational forces of the primary and secondary the object will be subjected to drag forces. The impact of these drag forces on the trajectory of these objects becomes more important as their size decreases. In the literature drag forces on such objects has been modeled using various functional form. Murray [11] used a simple drag force proportional to the particle velocity. On the other hand Burns and Jackson et al [1, 8] derived and used more general expression for this force which included radiation pressure and Poynting-Robertson drag. Elipe [2] considered this problem with "generalized" expressions for the drag forces. In this paper we model this force as one which is proportional to the square of the velocity and in its direction. The choice of this expression is motivated by the fact we want to consider the trajectories of these particles for short duration of time near the secondary. Under this assumption the prominent drag is provided by the solar wind and (possibly) the upper part of the planet atmosphere. Both of these contributions are modeled usually by a quadratic drag force. However our treatment actually applies with minor modifications whenever the size of the drag force can be expressed as a function of the particle position and velocity and its direction is along the particle velocity. We observe that with the inclusion of a drag force the system is energy dissipative. As a result this problem can not be treated by a Hamiltonian formalism.

It should noted however that if Earth is the secondary, additional forces due the J_2 effect and the moon have to be taken into consideration. Our treatment apply therefore in the generic cases where these forces are weaker or absent.

To put this problem within a broader context we note that for over a century the three body problem has been one of the outstanding problems in Celestial mechanics with continuous contributions from numerous authors. (We mention here only a few seminal references [9, 12, 13, 14, 15]. For an extensive list of contributions see [9, 14]. Recently however there has been additional interest in this difficult problem and some new results were derived for some very special cases [10, 3, 5]. Similarly the motion of satellites in the Earth atmosphere where they are subjected to the gravitational force of the Earth and a drag force had received wide attentions in the current literature [4, 7]. With this motivational background it seems justifiable to treat the three body problem with drag from a general mathematical and physical points of view.

It is our objective in this paper to consider the equations of the restricted three body problem with the addition of quadratic drag force. In this setting we derive reduced (approximate) equations for the trajectory of the object in a coordinates system centered at the secondary (planet) which is assumed to be in a general Keplerian orbit around the primary. The use of these coordinates (rather than barycentric coordinates) is due to their practical aspects as they obviate the need to "translate" the trajectory to a coordinate system attached to an observer on the planet. Thus these coordinates provide a natural framework to address the problem at hand. We shall show also that they lead to some major simplifications to the equations of motion and a new approximate expression for the Jacobi integral in three dimension three dimensions.

The plan of the paper is as follows: In Sec 2 we present the basic equations for the restricted three body problem with a quadratic drag term. The resulting equations of motion are linearized then under proper assumptions and recast in a coordinate system which is centered in the secondary. In Sec 3 these equations are simplified and reduced further. In Sec 4 we discuss adiabatic conservation laws and their relation to the Jacobi invariant [15]. Sec 5 considers some special cases and we derive simplified equations of motions and solutions under these assumptions. We also discuss the results of several simulations of the reduced equations and the impact of drag on the adiabatic invariants of motion. We end up in Sec 6 with summary and conclusions.

2. Basic Equations

In an inertial coordinate system whose origin is at the center of the central body (the primary) S we let $\mathbf{R}$, $\boldsymbol{\rho}$ denote the positions of the secondary and the third object respectively. (We assume that, approximately, the center of mass of the primary coincides with the center of mass of the three bodies. Actually, this assumption is not necessary but we make it to simplify the presentation) . Furthermore let $\mathbf{r}$ denote the relative position of the satellite with respect to the secondary. We then have (see $Fig\,1$) $\boldsymbol{\rho} = \mathbf{R} + \mathbf{r}$ and the equation of motion of the third body is given by

$$m_s\ddot{\boldsymbol{\rho}} = -\frac{GM_S m_s}{\rho^3}\boldsymbol{\rho} - \frac{GM_E m_s}{r^3}\mathbf{r} - m_s g(\alpha, \mathbf{r})(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^{\mathbf{1/2}}\dot{\mathbf{r}} \tag{1}$$

In this equation $\rho =\mid \boldsymbol{\rho} \mid$, $r =\mid \mathbf{r} \mid$, and M_S,M_E,m_s denote respectively the masses of the primary(Sun), the secondary and the third body. G is the constant of gravity and dots denote differentiation with respect to time. The last term in this equation represents the drag which is being modeled, as a quadratic function of the satellite velocity with respect to secondary and α is the drag coefficient (the factor m_s in front of g was added for convenience).

Throughout the paper we assume that $r \ll R$ i.e the third body is in the vicinity of the secondary. Obviously this assumption restricts the validity of our results . However as noted in the introduction when this assumption is invalid there are too many other perturbations that affect the trajectory of the third object.

Assuming that $r \ll R$ and using:

$$\rho = (\boldsymbol{\rho}\cdot\boldsymbol{\rho})^{1/2} = (\mathbf{R}+\mathbf{r},\ \mathbf{R}+\mathbf{r})^{1/2} = R\left[1 + \frac{2\mathbf{R}\cdot\mathbf{r}}{R^2} + \frac{r^2}{R^2}\right]^{1/2} \tag{2}$$

we can (by Taylor expansion) make the following approximation to eq. (1) [19]

$$\ddot{\mathbf{R}} + \ddot{\mathbf{r}} = \frac{-GM_S}{R^3}\left[\mathbf{R} + \mathbf{r} - \frac{3\mathbf{R}\cdot\mathbf{r}}{R^2}\mathbf{R}\right] - \frac{GM_E}{r^3}\mathbf{r} - g(\alpha, \mathbf{r})(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^{\mathbf{1/2}}\dot{\mathbf{r}}. \tag{3}$$

Using the fact that at any point

$$\ddot{\mathbf{R}} = -\frac{GM_S}{R^3}\mathbf{R} \tag{4}$$

eq. (3) reduces to

$$\ddot{\mathbf{r}} = -\frac{GM_S}{R^3}\left[\mathbf{r} - \frac{3\mathbf{R}\cdot\mathbf{r}}{R^2}\mathbf{R}\right] - \frac{GM_E}{r^3}\mathbf{r} - g(\alpha, \mathbf{r})(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^{\mathbf{1/2}}\dot{\mathbf{r}}. \tag{5}$$

Since the gravitational field of the third object is negligible it follows that the motion of the secondary is governed only by the gravitational field of S. As a result the motion of the secondary is in a fixed plane which we take as the $x-y$ plane. (Hence in polar coordinates the position of the secondary is given by (R,θ)). From the law of angular momentum conservation we have

$$M_E R^2\omega = L = \text{constant}. \tag{6}$$

where $\mathbf{\Omega}$ is the angular velocity vector and $\mathbf{\Omega} = \frac{d\theta}{dt}\mathbf{e}_z = (0,0,\omega)$. Using the relation (6) to eliminate R from eq. (5) we obtain

$$\ddot{\mathbf{r}} = -\frac{GM_S M_E^{3/2}}{L^{3/2}}\omega^{3/2}\left[\mathbf{r} - \frac{3\mathbf{R}\cdot\mathbf{r}}{R^2}\mathbf{R}\right] - \frac{GM_E}{r^3}\mathbf{r} - g(\alpha,\mathbf{r})(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^{\mathbf{1/2}}\dot{\mathbf{r}}. \tag{7}$$

In a coordinate system rotating with the secondary (and fixed at its center) eq. (7) becomes:

$$\begin{aligned}\ddot{\mathbf{r}} + 2\mathbf{\Omega}\times\mathbf{r} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) + \frac{d\mathbf{\Omega}}{dt}\times\mathbf{r} = k\Omega^{3/2}\left[\mathbf{r} - \frac{3\mathbf{R}\cdot\mathbf{r}}{R^2}\mathbf{R}\right] - \frac{GM_E}{r^3}\mathbf{r} \\ -g(\alpha,\mathbf{r})(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^{\mathbf{1/2}}(\dot{\mathbf{r}} + \mathbf{\Omega}\times\mathbf{r})\end{aligned} \tag{8}$$

where $k = \frac{GM_S M_E^{3/2}}{L^{3/2}}$. Here one should distinguish between $\mathbf{r}$ in eq. (7) which is a geocentric position vector with respect to an inertial frame while $\mathbf{r}$ in eq. (8) is the position vector in a rotating frame.

In this rotating system we now let the x-axis to be tangential but opposed to the motion of the secondary, the y-axis in the direction of $\mathbf{R}$ and the z-axis completes a right-handed system. (See $Fig\,2$). In this frame $\mathbf{r} = (x,y,z)$, $\mathbf{R} = (0,1,0)R$ and $\mathbf{\Omega} = (0,0,\dot{\theta}) = (0,0,\omega)$. In component form eq. (8) then becomes;

$$\ddot{x} - 2\omega\dot{y} - \omega^2 x - \dot{\omega}y = -k\omega^{3/2}x - \frac{GM_E x}{r^3} - g(\alpha,\mathbf{r})(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^{\mathbf{1/2}}(\dot{\mathbf{x}} - \omega\mathbf{y}) \tag{9}$$

$$\ddot{y} + 2\omega\dot{x} - \omega^2 y + \dot{\omega}x = -k\omega^{3/2}y + 3k\omega^{3/2}y - \frac{GM_E y}{r^3} - g(\alpha,\mathbf{r})(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^{\mathbf{1/2}}(\dot{\mathbf{y}} + \omega\mathbf{x}) \tag{10}$$

$$\ddot{z} = -k\omega^{3/2}z - \frac{GM_E z}{r^3} - g(\alpha,\mathbf{r})(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^{\mathbf{1/2}}\dot{\mathbf{z}} \tag{11}$$

We now perform a change of variables from t to $\theta(t)$ in these equations and divide the resulting equations by ω. This leads to

$$\omega x'' + \omega' x' = [\omega - k\omega^{1/2}]x + \omega' y + 2\omega y' - \frac{\beta x}{r^3} - g(\alpha,\mathbf{r})\omega(\mathbf{r}'\cdot\mathbf{r}')^{\mathbf{1/2}}[\mathbf{x}' - \mathbf{y}] \tag{12}$$

$$\omega y'' + \omega' y' = -\omega' x + [\omega + 2k\omega^{1/2}]y - 2\omega x' - \frac{\beta y}{r^3} - g(\alpha,\mathbf{r})\omega(\mathbf{r}'\cdot\mathbf{r}')^{\mathbf{1/2}}(\mathbf{y}' + \mathbf{x}) \tag{13}$$

$$\omega z'' + \omega' z' = -k\omega^{1/2}z - \frac{\beta z}{r^3} - g(\alpha,\mathbf{r})\omega(\mathbf{r}'\cdot\mathbf{r}')^{\mathbf{1/2}}\mathbf{z}' \tag{14}$$

where $\beta = \frac{GM_E}{\omega}$ and primes denote derivatives with respect to θ.

We now show that these equations can be simplified further by proper change of variables.

3. Derivation of the Reduced Equations

Since the secondary is in a Keplerian orbit around S we have [5]

$$\frac{1}{R} = C(1 + e\cos\theta), \quad \omega = \frac{LC^2}{M_E}(1 + e\cos\theta)^2 \tag{15}$$

where $C = \frac{GM_S M_E^2}{L^2}$ and e is the eccentricity of the orbit. (We place no restrictions on the value of e. However in our context the secondary is in elliptic orbit around the primary)

Substituting these expressions in eqs. (12)-(14) and dividing by $\frac{LC^2}{M_E}(1+e\cos\theta)$ yields after some algebra;

$$\frac{1}{(1+e\cos\theta)}[(1+e\cos\theta)^2 x']' = \left[(1+e\cos\theta) - \frac{kM_E^{1/2}}{CL^{1/2}}\right]x \tag{16}$$
$$+2[(1+e\cos\theta)y] - \frac{GM_E^3 x}{L^2C^4(1+e\cos\theta)^3 r^3} - g_1(\alpha,\mathbf{r},\theta)(\mathbf{r}'\cdot\mathbf{r}')^{\mathbf{1/2}}(\mathbf{x}'-\mathbf{y})$$

$$\frac{1}{(1+e\cos\theta)}[(1+e\cos\theta)^2 y']' = \left[(1+e\cos\theta) + \frac{2kM_E^{1/2}}{L^{1/2}C}\right]y \tag{17}$$
$$-2[(1+e\cos\theta)x]' - \frac{GM_E^3 y}{L^2C^4(1+e\cos\theta)^3 r^3} - g_1(\alpha,\mathbf{r},\theta)(\mathbf{r}'\cdot\mathbf{r}')^{1/2}(y'+x)$$

$$\frac{1}{(1+e\cos\theta)}[(1+e\cos\theta)^2 z']' = -\frac{kM_E^{1/2}}{L^{1/2}C}z - \frac{GM_E^3 z}{L^2C^4(1+e\cos\theta)r^3} - g_1(\alpha,\mathbf{r},\theta)(\mathbf{r}'\cdot\mathbf{r}')^{\mathbf{1/2}}\mathbf{z}' \tag{18}$$

Where $g_1(\alpha,\mathbf{r},\theta) = \frac{g(\alpha,\mathbf{r})\omega M_E}{LC^2(1+ecos\theta)}$.

We now introduce the variables

$$u = (1+e\cos\theta)x, \; v = (1+e\cos\theta)y, \quad w = (1+e\cos\theta)z \tag{19}$$

Substituting these variables in eqs. (16)-(18) leads to;

$$u'' = \frac{u}{1+e\cos\theta}\left[1 - \frac{kM_E^{1/2}}{CL^{1/2}}\right] + 2v' - \frac{Au}{(1+e\cos\theta)\sigma^3} - \tag{20}$$
$$g_2(\alpha,\mathbf{r},\theta)(\mathbf{r}'\cdot\mathbf{r}')^{1/2}\left[u' - v + \frac{ue\sin\theta}{(1+e\cos\theta)}\right]$$

$$v'' = \frac{v}{1+e\cos\theta}\left[1 + 2\frac{kM_E^{1/2}}{CL^{1/2}}\right] - 2u' - \frac{Av}{(1+e\cos\theta)\sigma^3} - \tag{21}$$
$$g_2(\alpha,\mathbf{r},\theta)(\mathbf{r}'\cdot\mathbf{r}')^{1/2}\left[v' + u + \frac{ve\sin\theta}{(1+e\cos\theta)}\right]$$

$$w'' = -w - \frac{Aw}{(1+e\cos\theta)\sigma^3} - g_2(\alpha,\mathbf{r},\theta)(\mathbf{r}'\cdot\mathbf{r}')^{1/2}\left[w' + \frac{we\sin\theta}{(1+e\cos\theta)}\right] \tag{22}$$

where $\boldsymbol{\sigma} = (u, v, w)$, $A = \frac{GM_E^3}{L^2C^4}$ and $g_2(\alpha, r, \theta) = \frac{g(\alpha,r)\omega M_E}{LC^2(1+e\cos\theta)^2}$. However it is now easy to see from the definitions of k, L, C and ω that

$$\frac{kM_E^{1/2}}{CL^{1/2}} = 1, \;\; g_2(\alpha, \mathbf{r}, \theta) = g(\alpha, \mathbf{r}) \tag{23}$$

and hence the equations of motion reduce to;

$$u'' = 2v' - \frac{Au}{(1+e\cos\theta)\sigma^3} - g(\alpha, \mathbf{r})(\mathbf{r}' \cdot \mathbf{r}')^{1/2}\left[u' - v + \frac{ue\sin\theta}{(1+e\cos\theta)}\right] \tag{24}$$

$$v'' = \frac{3v}{1+e\cos\theta} - 2u' - \frac{Av}{(1+e\cos\theta)\sigma^3} - g(\alpha, \mathbf{r})(\mathbf{r}' \cdot \mathbf{r}')^{1/2}\left[v' + u + \frac{ve\sin\theta}{(1+e\cos\theta)}\right] \tag{25}$$

$$w'' = -w - \frac{Aw}{(1+e\cos\theta)\sigma^3} - g(\alpha, \mathbf{r})(\mathbf{r}' \cdot \mathbf{r}')^{1/2}\left[w' + \frac{we\sin\theta}{(1+e\cos\theta)}\right] \tag{26}$$

Eqs. (24)-(26) represent a generalization of Hill's equations, to include eccentricity in the orbit of the secondary and quadratic drag force acting on the third object in the system subject to the approximation made in eq. (2). In the past the three body problem has been treated extensively in the literature under various approximations (See e.g.[3, 6]). However we believe that our derivation of these equations in rendezvous coordinates (and arbitrary eccentricity) near the secondary with drag did not appear in the literature so far.

When $g(\alpha, \mathbf{r}) = 0$ these equations resemble closely those that were derived for the rendezvous of a spacecraft and a satellite in [3, 5]. Furthermore in spite of the fact that these equations are nonlinear they are easily amenable to numerical computations. We observe also that if $A << 1$ and $g(\alpha, \mathbf{r}) << \mathbf{1}$ then it is straightforward to obtain approximate solutions to these equations by first order perturbation expansion. (Since the solution of these equations with $A = 0$ and $g(\alpha, \mathbf{r}) = \mathbf{0}$ is well known [7]).

4. Conservation Law

As we noted in the Introduction the addition of the drag term to the restricted three body problem renders this system to be dissipative. This precludes any exact conservation laws for the energy and angular momentum of the system. In spite of this fact it is useful (e.g as a check on the validity of numerical integration schemes) to derive equations for the evolution of these or related quantities under these circumstances. These equations are important for the treatment of weak drag forces and solutions of systems in three dimensions. (See next section).

To derive these "approximate conservation laws" for the angular momentum $\mathbf{J}$ and energy E that govern the motion of the third body we take the vector and scalar product of eq. (1) with $\boldsymbol{\rho}$ and $\dot{\boldsymbol{\rho}}$. After some simple algebraic manipulations we obtain

$$\frac{d\mathbf{J}}{dt} = \frac{d}{dt}(\boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}) = -\frac{GM_E}{r^3}\mathbf{R} \times \mathbf{r} - g(\alpha, \mathbf{r})(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^{1/2}(\mathbf{R} + \mathbf{r}) \times \dot{\mathbf{r}} \tag{27}$$

$$\frac{dE}{dt} = \frac{d}{dt}(\frac{\dot{\boldsymbol{\rho}}^2}{2} - \frac{GM_S}{\rho} - \frac{GM_E}{r}) = -\frac{GM_E}{r^3}\dot{\mathbf{R}} \cdot \mathbf{r} - g(\alpha, \mathbf{r})(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^{\mathbf{1/2}}(\dot{\boldsymbol{\rho}} \cdot \dot{\mathbf{r}}) \tag{28}$$

These two equations show clearly that in general neither the angular momentum nor the energy of the third body are constant.

To derive the corresponding equation for the rate of change of the energy (with respect to θ) from the reduced equations we multiply eqs. (24)-(26) by u', v', w' and sum. The result can be expressed in the form;

$$\left\{(\boldsymbol{\sigma}')^2 - \frac{3v^2 + \frac{2A}{\sigma}}{(1+e\cos\theta)} + w^2\right\}' = -\left[3v^2 + \frac{2A}{\sigma}\right]\frac{e\sin\theta}{(1+e\cos\theta)^2} - \tag{29}$$
$$2g(\alpha,\mathbf{r})(\mathbf{r}'\cdot\mathbf{r}')^{1/2}\left[(\boldsymbol{\sigma}')^2 + (uv'-vu') + \frac{e\sin\theta}{1+e\cos\theta}(\frac{\boldsymbol{\sigma}^2}{2})'\right]$$

We see from this representation that if $e = 0$ then the rate of change (with respect to θ) in the quantity on the left hand side of eq. (29) will be due solely to the dissipative effects of the drag. Furthermore if in addition $g(\alpha, \boldsymbol{\rho}) = 0$ then eq. (29) represents an exact conservation law. However as this conservation law was derived from the approximate equations of motion of the third body (i.e. eq. (5)) it is also an approximate invariant of the exact equations of motion. Eq. (29) presents therefore the modifications to the Jacobi integral when one considers the linearized restricted three body problem in three dimensions with quadratic drag near the secondary. When e and g are small the left hand side of eq. (29) can be considered as an "adiabatic invariant" of the motion.

We can derive an expression for the rate of change of a related quantity if we multiply eqs. (24),(25) by v and u respectively and subtract. This leads to;

$$\left[(uv'-vu') + (u^2+v^2)\right]' = \frac{3uv}{(1+e\cos\theta)} - g(\alpha,\mathbf{r})(\mathbf{r}'\cdot\mathbf{r}')^{1/2}[(uv'-vu') + (v^2+u^2)] \tag{30}$$

This equation will be important when one considers the restricted three body system in two dimension.

5. Some Special Cases

In this section we consider some special cases where the equations of motion of the third body can be reduced further. The presentation is in order of difficulty. From the simplest to the most general case.

5.1 Two Dimensional Case with $e = 0$ and $g(\alpha, r) = 0$

To begin with we consider the two dimensional case where all the three bodies are in $x - y$ plane, the secondary is in a circular orbit around the primary and there is no drag.

Under these conditions eq.(29) is an exact invariant and eqs.(29)- (30) reduce to a system of two differential equations for u, v.

$$(u')^2 + (v')^2 - 3v^2 - \frac{2A}{\sigma} = \text{constant}. \tag{31}$$

$$\left[(uv'-vu') + (u^2+v^2)\right]' = 3uv \tag{32}$$

To simplify these equations we introduce

$$u = \sigma \cos\phi, \quad v = \sigma \sin\phi \tag{33}$$

Eqs. (31),(32) then become

$$(\sigma')^2 + \sigma^2(\phi')^2 - \frac{2A}{\sigma} = 3\sigma^2 \sin^2\phi + c_1 \tag{34}$$

$$\left[\sigma^2(\phi'+1)\right]' = \frac{3}{2}\sigma^2 \sin(2\phi) \tag{35}$$

where c_1 is a constant. Many solutions of Hill's equations under these restrictions appeared in the literature [25,27]). We note that eqs. (34),(35) admit a consistent solution with

$$\sigma = constant, \quad (\phi')^2 = 3\sin^2\phi + c$$

where c is a constant. However this is not a valid solution of eqs. (24),(25) since under this assumptions eqs.(31),(32) are not independent. In fact it is well known that Hill's problem does not admit "circular" $\sigma = const.$ solution. However we can interpret this solution as one corresponding to the case where σ is almost constant viz $|\sigma'| \ll 1$.

We can reduce eqs. (34),(35) to a first order system by changing the independent variable from θ to ϕ and introduce $p = \frac{d\phi}{d\theta}$ as a new dependent variable. After some algebra we obtain;

$$p^2\left[(\frac{d\sigma}{d\phi})^2 + \sigma^2\right] - \frac{2A}{\sigma} = 3\sigma^2 \sin^2\phi + c_1 \tag{36}$$

$$p\frac{d}{d\phi}\left[\sigma^2(p+1)\right] = \frac{3}{2}\sigma^2 \sin(2\phi) \tag{37}$$

where c_1 is a constant.These equations provide a formulation for the orbit of the satellite in terms of it local coordinates with respect to the secondary only.

5.2 Two Dimensional Case with $e = 0$

Note that under these assumptions $\boldsymbol{\sigma} = \mathbf{r}$ and eqs. (29)-(30) reduce to

$$\left[(u')^2 + (v')^2 - 3v^2 - \frac{2A}{\sigma}\right]' = -2g(\alpha,\sigma)(\boldsymbol{\sigma}'\cdot\boldsymbol{\sigma}')^{1/2}\left[(uv'-vu') + (u^2+v^2)\right] \tag{38}$$

$$\left[(uv'-vu') + (u^2+v^2)\right]' = 3uv - g(\alpha,\sigma)(\boldsymbol{\sigma}'\cdot\boldsymbol{\sigma}')^{1/2}\left[(uv'-vu') + (u^2+v^2)\right] \tag{39}$$

Using eq. (33) we obtain

$$\left[(\sigma')^2 + \sigma^2(\phi')^2 - 3\sigma^2 sin^2(\phi) - \frac{2A}{\sigma}\right]' = -2g(\alpha,\boldsymbol{\sigma})(\boldsymbol{\sigma}'\cdot\boldsymbol{\sigma}')^{1/2}\sigma^2(\phi'+1) \tag{40}$$

$$\left[\sigma^2(\phi'+1)\right]' = \frac{3}{2}\sigma^2\sin(2\phi) - g(\alpha,\boldsymbol{\sigma})(\boldsymbol{\sigma}'\cdot\boldsymbol{\sigma}')^{1/2}\left[\sigma^2(\phi'+1)\right] \tag{41}$$

We can rewrite eq.(41) in the form

$$\frac{(\sigma^2)'}{\sigma^2} = \frac{\frac{3}{2}\sin(2\phi) - \phi''}{\phi'+1} - g(\alpha,\boldsymbol{\sigma})(\boldsymbol{\sigma}'\cdot\boldsymbol{\sigma}')^{1/2} \tag{42}$$

This equation demonstrates the role that the drag term can play in many cases, especially when the first term on the right hand side of this equation is small. Under these circumstances the drag term can determine whether σ^2 (that is the distance between the secondary and the third object in the system) will increase or decrease and as a result change the long term nature of the third object trajectory.

Eliminating the drag forces from (41) and (40) we obtain

$$\left[(\sigma')^2 + \sigma^2(\phi')^2 - \frac{2A}{\sigma} - 2\sigma^2(\phi' + 1)\right]' = 6\sigma\sigma' \sin^2\phi \tag{43}$$

This implies that if $|\sigma'| \ll 1$ (that is the orbit of the third body is almost circular) then the left hand side represents an adiabatic invariant of the system.

5.3 Three Dimensional Case with $e \neq 0$

The relevant equations in this case are (24)-(26). Due to their complexity we can solve these (coupled) equations analytically only through the use of first order perturbation expansion. To this end we observe that from a practical point of view both A and g are small. Therefore we write

$$u = u_0 + \epsilon u_1, \;\; v = v_0 + \epsilon v_1, \;\; w = w_0 + \epsilon w_1. \tag{44}$$

Substituting these in (24)-(26) and neglecting terms containing $A\epsilon$ and $g(\alpha, r)\epsilon$ as being of second order we obtain to the zeroth order in ϵ

$$u_0'' = 2v_0', \;\; v_0'' = \frac{3v_0}{1 + e\cos\theta} - 2u_0', \;\; w_0'' = -w_0. \tag{45}$$

A solution of the (resulting) equation for v is

$$v(\theta) = C\sin\theta\,(1 + e\cos\theta) \tag{46}$$

Using this solution it is possible to write down explicitly the general solution to (45) (see *Ref* 16 for details). The first order equations in ϵ are the same as (24)-(26) except that in the terms containing A and g,the variables u, v, w should be replaced by u_0, v_0, w_0 while in the other terms these variables should be replaced by u_1, v_1, w_1. The general solution of this system of equations can be done then using standard methods for the solution of inhomogeneous system of linear equations and will not be presented here for brevity.

5.4 Model Validation

The equations of motion for the third body that were derived in the previous sections are highly nonlinear and therefore we have to resort to numerical methods to gauge their validity. (Actually it is well known that the three body problem, even without drag, has only few exact analytical solutions in very special cases [5,6]).

To evaluate the effect of the drag term on the trajectory of the third object in the system we simulated (for comparison purposes, using MATLAB) eqs. (24), (25) in two dimensions with $e = 0$ and $A = 1$. The initial values used were $u(0) = 15, v(0) =$

$0, u'(0) = 0$ and $v'(0) = 0$. For this setting the values of g were varied from $g = 0$ to $g = 10^{-4}$ in steps of 2.5×10^5 (the relative error in each integration step was set to 10^{-8}) . $Fig\,3.$ presents the difference in the values of σ for the four orbits with drag from its value for the orbit without drag.

For each of these orbits we computed also the left hand side of eq.(38) along the orbit viz.

$$J = (\boldsymbol{\sigma}')^2 - 3v^2 - \frac{2A}{\sigma} \tag{47}$$

$Fig\,4.$ shows the difference between the values of J along the orbits with drag from its value for the orbit without drag. (For the orbit without drag J is a conserved quantity and its value in this case is -0.1333.) We note the similarity between $Figs\,3, 4$ although they have different scales.

We compared also the (long term) trajectories predicted by our approximate equations eqs (24)-(26) with those that are obtained by direct numerical simulation of the exact equations of motion (1) (with $g = 0$). To this end we assumed that the secondary is in circular orbit around the primary with $R = 1AU$. The initial conditions on the third object correspond to those that will let this object stay in circular orbit around the secondary if the primary was absent with $r = 0.002AU$. In these computations we set the step error tolerance to 10^{-12} for a total of 10^5 time steps. The difference in the values of r along the trajectories that follows from these two sets of equations for $100days$ is presented in $Fig\,5$. As expected this difference grows with time but the error throughout this time period remains bounded by $3 \times 10^{-7}AU$. This confirms the accuracy of eqs. (24)-(26).

Summary and Conclusions

In this paper we discussed the three body problem under two approximations. The first approximation results from neglecting the influence of the third body on the trajectory of the secondary. The second is due to the linearization of the expression for the gravitational force under the assumption $r \ll R$. This approximation becomes asymptotically exact as $r/R \to 0$ i.e. we are assuming that the third object is close to the secondary. This is exactly the physical configuration of the system we wish to consider. However these equations and approximations will obviously become invalid when these assumptions are violated.

The results of this paper show that due to an "accidental" coincidence among the various constants that govern the restricted three body system (in three dimensions) the approximate equations of motion can be simplified analytically. As we pointed out in the last section this leads to further simplifications in some limiting cases. An adiabatic conservation law was derived for these equations which can be considered as a natural extension of the (exact) Jacobi integral for this problem in two dimensions. We demonstrated also that in some cases the trajectory of the third body can be computed by solving a reduced system of two nonlinear first order differential equations.

From another practical point of view the existence of many small objects in the solar system is still unknown. To gauge their potential for a possible collision (or fly by) with a planet it is important to include all known sources that might impact their trajectory

(as they near the planet). In this paper we derived proper equations for the inclusion of drag forces on the trajectory of these objects. We demonstrated also (analytically and by simulation) that these effects are important for the computation of the correct trajectory for these objects.

Finally we note in passing that a similar treatment can be made when the drag forces are linear in the third object velocity which might be important in some celestial contexts. A complete treatment of this case is available from the author.

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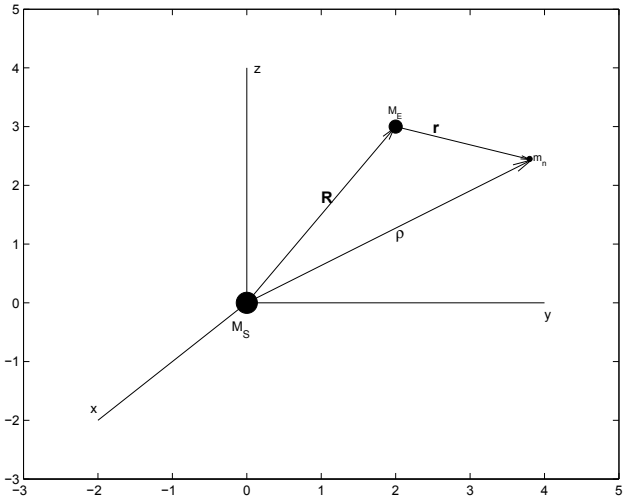

Fig. 1 A diagrammatic presentation of the three body problem discussed in this paper. We assume that $m_n \ll M_E \ll M_S$ and $r \ll R$.

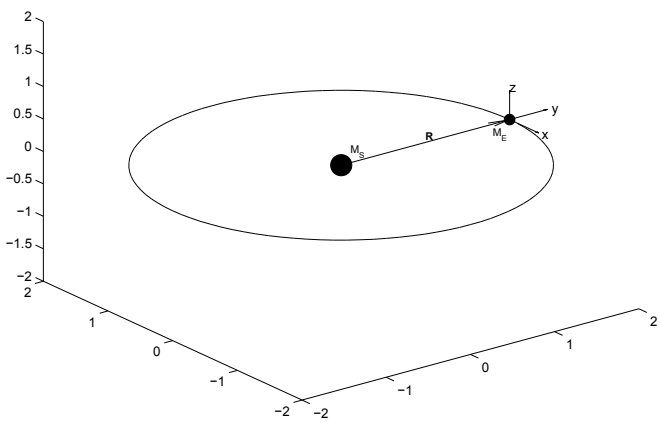

Fig. 2 The frame attached to M_E which is rotating anti-clockwise around M_S

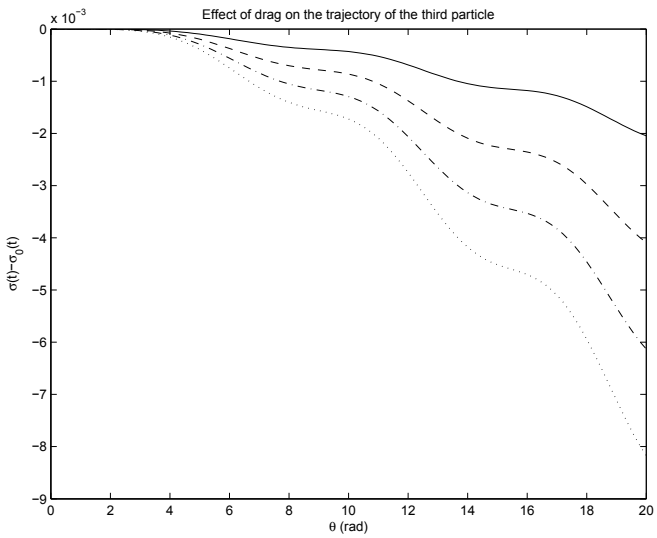

Fig. 3 The differences $\sigma(t)-\sigma_0(t)$ for the trajectories with drag from $\sigma_0(t)$ which is the trajectory without drag with $\sigma_0(0) = 15$. All trajectories have the same initial. condition. The values of g used were $2.5 \times 10^{-5}, 5 \times 10^{-5}, 7.5 \times 10^{-5}$ and 10^{-4} These are represented respectively, by the solid,dashes,dashed-dot and dotted lines.

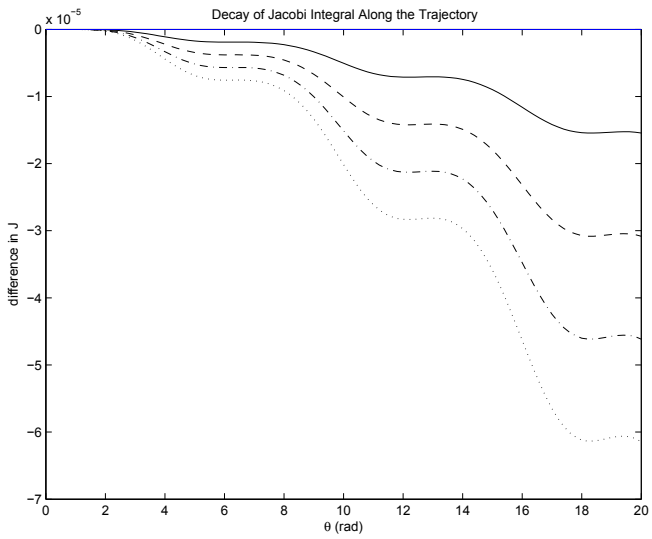

Fig. 4 The difference in the values of J for trajectories with drag from the one without drag. The constant value of J along the trajectory without drag is represented by the zero line. (The actual value of J in this case is -0.1333). Same values for g as in $Fig\,3$.

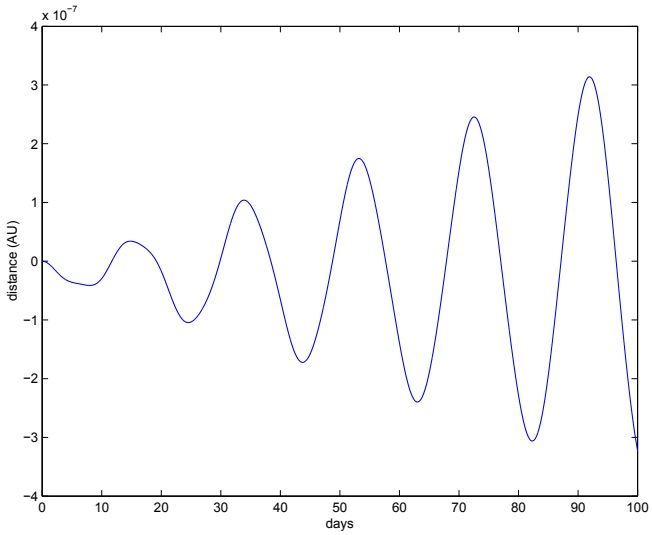

Fig. 5 The difference in the values of $r(t)$ which are obtained from the numerical integration of eq. (1) and eqs. (24)-(26).