EJTP 7, No. 23 (2010) 161–172 *Electronic Journal of Theoretical Physics*

Relativistic Spin Operator with Observers in Motion

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Received 28 January 2010, Accepted 20 February 2010, Published 20 March 2010

Abstract: We obtain transformation equations for the Bell basis states under an arbitrary Lorentz boost and compute the expectation values of the relativistic center of mass spin operator under each of these boosted states. We also obtain expectation values for spin projections along the axes.

Keywords: Quantum Information; Entanglement; Bell Basis; Lorentz Transformations
PACS (2008): 03.67.-a; 03.65.Ud; 03.67.Bg; 03.30.+p; 03.65.-w

1. Introduction

Ever since the publication of the "EPR Paradox" as exposed by Einstein, Podolsky & Rosen in their seminal article [1], "quantum entanglement" has been mystifying physicists across the world. Even to this day, a complete explanation of this phenomenon defies human endeavors although its existence is, now, universally acknowledged. In fact, the focus of research has perceptibly shifted to exploiting this unique property of composite quantum systems for performing information processing tasks with unprecedented efficiency – "quantum entanglement" is gradually being acknowledged as a resource for "quantum computing and communication". It, has, therefore, become all the more necessary to examine the effect of "relativistic transformations" on mutually entangled subsystems. This essentially is the objective of this work. We obtain transformation equations for the Bell basis states under an arbitrary boost and compute the expectation values of the relativistic center of mass spin operator under each of these boosted states. Our work is based on Bohm's version of the paradox [2,3] applying discrete spin states.

Pioneering work in this direction was initiated by Czachor [4]. However, the paper confines itself to the averaging of the relativistic spin operator [5] (that relates to the

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spatial components of the Pauli Lubanski vector) over the spin singlet state in the laboratory frame. The work does not consider the boosting of the Bell states and hence, does not embrace the case of relative motion of an arbitrary observer frame. The paper does, however, establish that relativistic effects are seriously relevant to quantum computing operations. This was followed by a spurt of papers [6-8]. Each of these works, in essence, takes a certain specific case and examines its physical implications instead of solving the problem in its generality. For instance, boosts are always taken perpendicular to the momenta, Alsing [6] investigates the massless case in a specific gauge formulation wherein the photon polarization vectors are coplanar with the electric and magnetic fields, Gingrich [8] assumes a Gaussian momentum distribution, Pachos [7] looks at a relativistic formulation on the basis of a magnetic dipole-dipole interaction etc. The important point is that none of these papers attempts an averaging of the center of mass relativistic spin operator with boosted Bell states which could lead to relativistic correction in the Bell's inequalities for observers in arbitrary motion. Following exactly the same methodology as Alsing [6] i.e. using Wigner rotation & momentum eigenstates Terashima & Ueda [9-10] show that the spin singlet state undergoes a change in anti-correlation due to Lorentz transformation. However, while Alsing [6] represents the state in terms of 4 – component Dirac spinors, Terashima & Ueda [9-10] use Hilbert space state vectors. While they do consider boosted states to accommodate observer frame, they do not use the center of mass relativistic spin operator. The spin observable considered in [11] is different from the center of mass relativistic spin operator. Without explicitly introducing a functional form of the spin operator, they define it as an operator having the property "based on the sameness of the expectation values of one particle spin measurement evaluated in two relative reference frames, one in the laboratory frame in which the particle has a velocity ˜v and the observer is at rest, the other in the moving frame Lorentz boosted with ˜v, in which the particle is at rest and the observer is moving with a velocity -˜v." However, the appropriateness of such an operator for a two/multi particle state is still an open question.

2. The Relativistic Spin Operator [5]

The relativistic spin operator, S, has been an issue of considerable debate among physicists. The fundamental properties that need to be satisfied by an acceptable spin operator, S, are [5]:-

(a) It must be expressible as some combination of the complete set of generators of the Poincare group i.e. $P^0, \mathbf{P}, \mathbf{J}, \mathbf{K}$;

(b) It must behave as a 3-vector under spatial rotations i.e. $[S^i, J^j] = i\varepsilon^{ijk}S^k$ whence it must be linear in the generators $\mathbf{J}, \mathbf{K}$ so that we can express it as $\mathbf{S} = f_1\mathbf{J} + f_2\mathbf{K} + f_3\mathbf{P} + f_4\left(\mathbf{P}.\mathbf{J}\right)P + f_5\left(\mathbf{P}.\mathbf{K}\right)P + f_6\left(\mathbf{P}\times\mathbf{J}\right) + f_7\left(\mathbf{P}\times\mathbf{K}\right)$

where each $f_i \equiv f_i\left(\mathbf{P}^2, P^0\right)$;

(c) It should be independent of generators of translations i.e. $[S^i, P^\mu] = 0$

(d) It should be equal to the difference of the total angular momentum and the orbital angular momentum i.e. $\mathbf{S} = \mathbf{J} - \mathbf{x} \times \mathbf{P}$;

(e) Being an angular momentum, it must satisfy the commutators $[S^i, S^j] = i\varepsilon^{ijk}S^k$.

Application of (a) –(e) yield [5]:-

$$\mathbf{S} = \frac{P^0}{M}\mathbf{J} - \frac{1}{M}(\mathbf{P} \times \mathbf{K}) - \frac{1}{M(M+P^0)}\mathbf{P}(\mathbf{P}.\mathbf{J}) \tag{1}$$

The above expression for the relativistic spin operator is, in fact, the value of the "total angular momentum" operator of the particle in its rest system. This is easily seen. The relativistic transformation of an arbitrary 3-vector that has the value x in a system with momentum P to its rest frame is given by $\mathbf{x}' = \mathbf{x} - \mathbf{P}\frac{\mathbf{x}.\mathbf{P}+Mx^0}{M(M+P^0)}$ whence, the expression for the spatial part of the Pauli Lubanski operator $W^\mu = \frac{1}{2}\varepsilon^{\alpha\beta\chi\mu}J_{\alpha\beta}P_\chi$ $(\mathbf{W} = P^0\mathbf{J} - \mathbf{P} \times \mathbf{K}, W^0 = (\mathbf{P}.\mathbf{J}))$ in its rest frame becomes $\mathbf{W}' = P^0\mathbf{J} - \mathbf{P} \times \mathbf{K} - \mathbf{P}\frac{(\mathbf{P}.\mathbf{J})}{(M+P^0)}$. Defining the relativistic spin operator $\mathbf{S}$ by $P^0\mathbf{J}' = M\mathbf{J}' = \mathbf{W}' = M\mathbf{S}$, we obtain the above expression for $\mathbf{S}$.

In the canonical representation of the Poincare group characterized by the momentum operator being diagonal i.e. $P^\mu = p^\mu$ and the infinitesimal generators of rotations and boosts taking the form $\mathbf{J} = -i\mathbf{p} \times \nabla_p + \mathbf{S}$ and $\mathbf{K} = ip^0\nabla_p + \frac{\mathbf{p}\times\mathbf{S}}{(M+p^0)}$ [12], we have $(\mathbf{P}.\mathbf{J}) = 0$ whence $\mathbf{S} = \frac{P^0}{M}\mathbf{J} - \frac{1}{M}(\mathbf{P} \times \mathbf{K}) = \frac{\mathbf{W}}{M}$. In this representation, one can set $\mathbf{S} = \mathbf{s} = \frac{1}{2}\sigma$ being the two dimensional representation $D^{(1/2)}$ of the little group $SO(3)$ in the rest frame. Identifying $\frac{P^0}{M}\mathbf{J}$ and $-\frac{1}{M}(\mathbf{P} \times \mathbf{K})$ respectively as being the components of spin parallel and perpendicular to direction of momentum $\mathbf{p}$, and using the above explicit representation of $\mathbf{J}, \mathbf{K}$, we obtain

$$\mathbf{S} = \mathbf{S}_\parallel + \mathbf{S}_\perp = \frac{p^0}{M|\mathbf{p}|^2}(\mathbf{p}.\mathbf{s})\,\mathbf{p} + \left(\mathbf{s} - \frac{1}{|\mathbf{p}|^2}(\mathbf{p}.\mathbf{s})\,\mathbf{p}\right) \tag{2}$$

(In the rest frame, obviously, $\mathbf{S}_\parallel = \mathbf{s}_\parallel = \frac{1}{|\mathbf{p}|^2}(\mathbf{p}.\mathbf{s})\,\mathbf{p}$).

3. The Lorentz Transformation of Single Particle States

The effect of Lorentz boosts on single particle states is well documented and can be expressed in terms of the Wigner rotation $R_W(p, \Lambda)$ [13] as:

$$U(\Lambda)|p, s\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma=-s}^{s} D^j_{\sigma s}(R_W(\Lambda, \mathbf{p}))\,|\Lambda\mathbf{p}, \sigma\rangle \tag{3}$$

Explicit expression for the Wigner rotation has been obtained by various approaches. Following Halpern [14], we can, making use of the $SL(2, \mathbb{C})$ representation of the Lorentz group, obtain an explicit representation of $R_W(\mathbf{p}, \Lambda)$ for spin massive particles as:

$$D^{1/2}(R_W(p, \Lambda)) = \cos\left(\frac{\theta}{2}\right) + i(\sigma.n)\sin\left(\frac{\theta}{2}\right) \tag{4}$$

where

$$\cos\left(\frac{\theta}{2}\right)=\frac{\cosh\left(\frac{\alpha}{2}\right)\cosh\left(\frac{\beta}{2}\right)+(e.f)\sinh\left(\frac{\alpha}{2}\right)\sinh\left(\frac{\beta}{2}\right)}{\left[\frac{1}{2}+\cosh\left(\frac{\alpha}{2}\right)\cosh\left(\frac{\beta}{2}\right)+(e.f)\sinh\left(\frac{\alpha}{2}\right)\sinh\left(\frac{\beta}{2}\right)\right]^{1/2}} \tag{5}$$

$$n\sin\left(\frac{\theta}{2}\right)=\frac{(e\times f)\sinh\left(\frac{\alpha}{2}\right)\sinh\left(\frac{\beta}{2}\right)}{\left[\frac{1}{2}+\cosh\left(\frac{\alpha}{2}\right)\cosh\left(\frac{\beta}{2}\right)+(e.f)\sinh\left(\frac{\alpha}{2}\right)\sinh\left(\frac{\beta}{2}\right)\right]^{1/2}} \tag{6}$$

$\cosh\alpha=\Lambda_0^0=\gamma=(1-v_b^2)^{-1/2}$, $\cosh\beta=\frac{p^0}{M}$, $f=\frac{p}{|p|}$ and e is the unit vector in the direction of the boost Λ.

4. The Lorentz Transformation of Bell States

The Bell state basis for a pair of spin particles with respective momenta $p, -p$ in the rest frame consists of the basis vectors
$|\Phi^+\rangle=\frac{1}{\sqrt{2}}\left(\left|p,\frac{1}{2}\right\rangle\otimes\left|-p,\frac{1}{2}\right\rangle+\left|p,-\frac{1}{2}\right\rangle\otimes\left|-p,-\frac{1}{2}\right\rangle\right)$,
$|\Phi^-\rangle=\frac{1}{\sqrt{2}}\left(\left|p,\frac{1}{2}\right\rangle\otimes\left|-p,\frac{1}{2}\right\rangle-\left|p,-\frac{1}{2}\right\rangle\otimes\left|-p,-\frac{1}{2}\right\rangle\right)$,
$|\Psi^+\rangle=\frac{1}{\sqrt{2}}\left(\left|p,\frac{1}{2}\right\rangle\otimes\left|-p,-\frac{1}{2}\right\rangle+\left|p,-\frac{1}{2}\right\rangle\otimes\left|-p,\frac{1}{2}\right\rangle\right)$
and $|\Psi^-\rangle=\frac{1}{\sqrt{2}}\left(\left|p,\frac{1}{2}\right\rangle\otimes\left|-p,-\frac{1}{2}\right\rangle-\left|p,-\frac{1}{2}\right\rangle\otimes\left|-p,\frac{1}{2}\right\rangle\right)$.
The Lorentz transformation induced by Λ transforms these basis states as follows:

(i) $|\Phi^+\rangle$ transforms as:

$$U(\Lambda)\left|\Phi^+\right\rangle=\frac{1}{\sqrt{2}}\sum_{\sigma,\sigma'=-1/2}^{1/2}\left\{\begin{array}{l}\sqrt{\frac{(\Lambda p)^0}{p^0}}D^{(1/2)}_{\sigma,1/2}\left(R_W(\Lambda,p)\right)\sqrt{\frac{(\Lambda(-p,p^0))^0}{p^0}}D^{(1/2)}_{\sigma',1/2}\left(R_W\left(\Lambda,(-p,p^0)\right)\right)|\Lambda p,\sigma\rangle\otimes|-\Lambda p,\sigma'\rangle\\+\sqrt{\frac{(\Lambda p)^0}{p^0}}D^{(1/2)}_{\sigma,-1/2}\left(R_W(\Lambda,p)\right)\sqrt{\frac{(\Lambda(-p,p^0))^0}{p^0}}D^{(1/2)}_{\sigma',-1/2}\left(R_W\left(\Lambda,(-p,p^0)\right)\right)|\Lambda p,\sigma\rangle\otimes|-\Lambda p,\sigma'\rangle\end{array}\right\}$$

Now, making use of

$$D^{1/2}\left(R_W(p,\Lambda)\right)=\cos\left(\frac{\theta}{2}\right)+i(\sigma.n)\sin\left(\frac{\theta}{2}\right)=\begin{pmatrix}\cos\frac{\theta}{2}+in_3\sin\frac{\theta}{2} & (in_1+n_2)\sin\frac{\theta}{2}\\(in_1-n_2)\sin\frac{\theta}{2} & \cos\frac{\theta}{2}-in_3\sin\frac{\theta}{2}\end{pmatrix}$$

$$D^{1/2}\left(R_W\left((-p,p^0),\Lambda\right)\right)=\cos\left(\frac{\theta}{2}\right)-i(\sigma.n)\sin\left(\frac{\theta}{2}\right)=\begin{pmatrix}\cos\frac{\theta}{2}-in_3\sin\frac{\theta}{2} & -(in_1+n_2)\sin\frac{\theta}{2}\\-(in_1-n_2)\sin\frac{\theta}{2} & \cos\frac{\theta}{2}+in_3\sin\frac{\theta}{2}\end{pmatrix}$$

we obtain

$$U(\Lambda)\left|\Phi^+\right\rangle=\frac{1}{\sqrt{2}}\frac{(\Lambda p)^0}{p^0}\left\{\begin{array}{l}\left[1-2n_2(n_2+in_1)\sin^2\frac{\theta}{2}\right]\left|\Lambda p,-\frac{1}{2}\right\rangle\otimes\left|-\Lambda p,-\frac{1}{2}\right\rangle-\left[2n_2\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}-in_3\sin\frac{\theta}{2}\right)\right]\left|\Lambda p,\frac{1}{2}\right\rangle\otimes\left|-\Lambda p,-\frac{1}{2}\right\rangle\\+\left[2n_2\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}+in_3\sin\frac{\theta}{2}\right)\right]\left|\Lambda p,-\frac{1}{2}\right\rangle\otimes\left|-\Lambda p,\frac{1}{2}\right\rangle+\left[1-2n_2(n_2-in_1)\sin^2\frac{\theta}{2}\right]\left|\Lambda p,\frac{1}{2}\right\rangle\otimes\left|-\Lambda p,\frac{1}{2}\right\rangle\end{array}\right.$$

$$=\frac{1}{\sqrt{2}}\frac{(\Lambda p)^0}{p^0}\left\{\begin{array}{l}\left[1-n_2^2(1-\cos\theta)\right]\left(\left|\Lambda p,-\frac{1}{2}\right\rangle\otimes\left|-\Lambda p,-\frac{1}{2}\right\rangle+\left|\Lambda p,\frac{1}{2}\right\rangle\otimes\left|-\Lambda p,\frac{1}{2}\right\rangle\right)\\-in_1n_2(1-\cos\theta)\left(\left|\Lambda p,-\frac{1}{2}\right\rangle\otimes\left|-\Lambda p,-\frac{1}{2}\right\rangle-\left|\Lambda p,\frac{1}{2}\right\rangle\otimes\left|-\Lambda p,\frac{1}{2}\right\rangle\right)\\-n_2\sin\theta\left(\left|\Lambda p,\frac{1}{2}\right\rangle\otimes\left|-\Lambda p,-\frac{1}{2}\right\rangle-\left|\Lambda p,-\frac{1}{2}\right\rangle\otimes\left|-\Lambda p,\frac{1}{2}\right\rangle\right)\\+in_2n_3(1-\cos\theta)\left(\left|\Lambda p,\frac{1}{2}\right\rangle\otimes\left|-\Lambda p,-\frac{1}{2}\right\rangle+\left|\Lambda p,-\frac{1}{2}\right\rangle\otimes\left|-\Lambda p,\frac{1}{2}\right\rangle\right)\end{array}\right\}$$

$$= \frac{(\Lambda p)^0}{p^0} \left\{ \left[1 - n_2^2 (1-\cos\theta)\right] \left|\Phi_\Lambda^+\right\rangle + in_1 n_2 (1-\cos\theta) \left|\Phi_\Lambda^-\right\rangle - n_2 \sin\theta \left|\Psi_\Lambda^-\right\rangle + in_2 n_3 (1-\cos\theta) \left|\Psi_\Lambda^+\right\rangle \right\} \quad (7)$$

(ii) $|\Phi^-\rangle$ transforms as:

$$U(\Lambda)\left|\Phi^-\right\rangle = \frac{1}{\sqrt{2}} \sum_{\sigma,\sigma'=-1/2}^{1/2} \left\{ \begin{array}{l} \sqrt{\frac{(\Lambda p)^0}{p^0}} D^{(1/2)}_{\sigma,1/2}(R_W(\Lambda,p)) \sqrt{\frac{(\Lambda(-p,p^0))^0}{p^0}} D^{(1/2)}_{\sigma',1/2}\left(R_W\left(\Lambda,(-p,p^0)\right)\right) |\Lambda p,\sigma\rangle \otimes |-\Lambda p,\sigma'\rangle \\ -\sqrt{\frac{(\Lambda p)^0}{p^0}} D^{(1/2)}_{\sigma,-1/2}(R_W(\Lambda,p)) \sqrt{\frac{(\Lambda(-p,p^0))^0}{p^0}} D^{(1/2)}_{\sigma',-1/2}\left(R_W\left(\Lambda,(-p,p^0)\right)\right) |\Lambda p,\sigma\rangle \otimes |-\Lambda p,\sigma'\rangle \end{array} \right\}$$

$$= \frac{1}{\sqrt{2}} \frac{(\Lambda p)^0}{p^0} \left\{ \begin{array}{l} \left[n_1(n_1 - in_2)(1-\cos\theta) - 1\right] \left|\Lambda p, -\tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, -\tfrac{1}{2}\right\rangle - \left[in_1 \sin\theta + n_1 n_3 (1-\cos\theta)\right] \left|\Lambda p, \tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, -\tfrac{1}{2}\right\rangle \\ + \left[in_1 \sin\theta - n_1 n_3 (1-\cos\theta)\right] \left|\Lambda p, -\tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, \tfrac{1}{2}\right\rangle + \left[1 - n_1(n_1 + in_2)(1-\cos\theta)\right] \left|\Lambda p, \tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, \tfrac{1}{2}\right\rangle \end{array} \right\}$$

$$= \frac{1}{\sqrt{2}} \frac{(\Lambda p)^0}{p^0} \left\{ \begin{array}{l} -\left[1 - n_1^2(1-\cos\theta)\right] \left(\left|\Lambda p, -\tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, -\tfrac{1}{2}\right\rangle - \left|\Lambda p, \tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, \tfrac{1}{2}\right\rangle\right) \\ -in_1 n_2 (1-\cos\theta) \left(\left|\Lambda p, -\tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, -\tfrac{1}{2}\right\rangle + \left|\Lambda p, \tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, \tfrac{1}{2}\right\rangle\right) \\ -in_1 \sin\theta \left(\left|\Lambda p, \tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, -\tfrac{1}{2}\right\rangle - \left|\Lambda p, -\tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, \tfrac{1}{2}\right\rangle\right) \\ -n_1 n_3 (1-\cos\theta) \left(\left|\Lambda p, \tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, -\tfrac{1}{2}\right\rangle + \left|\Lambda p, -\tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, \tfrac{1}{2}\right\rangle\right) \end{array} \right\}$$

$$= \frac{(\Lambda p)^0}{p^0} \left\{ -in_1 n_2 (1-\cos\theta) \left|\Phi_\Lambda^+\right\rangle + \left[1 - n_1^2 (1-\cos\theta)\right] \left|\Phi_\Lambda^-\right\rangle - in_1 \sin\theta \left|\Psi_\Lambda^-\right\rangle - n_1 n_3 (1-\cos\theta) \left|\Psi_\Lambda^+\right\rangle \right\} \quad (8)$$

(iii) For $|\Psi^+\rangle$, we have

$$U(\Lambda)\left|\Psi^+\right\rangle = \frac{1}{\sqrt{2}} \sum_{\sigma,\sigma'=-1/2}^{1/2} \left\{ \begin{array}{l} \sqrt{\frac{(\Lambda p)^0}{p^0}} D^{(1/2)}_{\sigma,1/2}(R_W(\Lambda,p)) \sqrt{\frac{(\Lambda(-p,p^0))^0}{p^0}} D^{(1/2)}_{\sigma',-1/2}\left(R_W\left(\Lambda,(-p,p^0)\right)\right) |\Lambda p,\sigma\rangle \otimes |-\Lambda p,\sigma'\rangle \\ +\sqrt{\frac{(\Lambda p)^0}{p^0}} D^{(1/2)}_{\sigma,-1/2}(R_W(\Lambda,p)) \sqrt{\frac{(\Lambda(-p,p^0))^0}{p^0}} D^{(1/2)}_{\sigma',1/2}\left(R_W\left(\Lambda,(-p,p^0)\right)\right) |\Lambda p,\sigma\rangle \otimes |-\Lambda p,\sigma'\rangle \end{array} \right\}$$

$$= \frac{1}{\sqrt{2}} \frac{(\Lambda p)^0}{p^0} \left\{ \begin{array}{l} \left[n_3(n_1 - in_2)(1-\cos\theta)\right] \left|\Lambda p, -\tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, -\tfrac{1}{2}\right\rangle + \left[1 - in_3 \sin\theta - n_3^2 (1-\cos\theta)\right] \left|\Lambda p, \tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, -\tfrac{1}{2}\right\rangle \\ + \left[1 + in_3 \sin\theta - n_3^2 (1-\cos\theta)\right] \left|\Lambda p, -\tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, \tfrac{1}{2}\right\rangle - \left[n_3(n_1 + in_2)(1-\cos\theta)\right] \left|\Lambda p, \tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, \tfrac{1}{2}\right\rangle \end{array} \right\}$$

$$= \frac{1}{\sqrt{2}} \frac{(\Lambda p)^0}{p^0} \left\{ \begin{array}{l} -in_2 n_3 (1-\cos\theta) \left(\left|\Lambda p, -\tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, -\tfrac{1}{2}\right\rangle + \left|\Lambda p, \tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, \tfrac{1}{2}\right\rangle\right) \\ +n_1 n_3 (1-\cos\theta) \left(\left|\Lambda p, -\tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, -\tfrac{1}{2}\right\rangle - \left|\Lambda p, \tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, \tfrac{1}{2}\right\rangle\right) \\ + \left[1 - n_3^2 (1-\cos\theta)\right] \left(\left|\Lambda p, \tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, -\tfrac{1}{2}\right\rangle + \left|\Lambda p, -\tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, \tfrac{1}{2}\right\rangle\right) - \\ in_3 \sin\theta \left(\left|\Lambda p, \tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, -\tfrac{1}{2}\right\rangle - \left|\Lambda p, -\tfrac{1}{2}\right\rangle \otimes \left|-\Lambda p, \tfrac{1}{2}\right\rangle\right) \end{array} \right\}$$

$$= \frac{(\Lambda p)^0}{p^0} \left\{ -in_2 n_3 (1-\cos\theta) \left|\Phi_\Lambda^+\right\rangle - n_1 n_3 (1-\cos\theta) \left|\Phi_\Lambda^-\right\rangle + \left[1 - n_3^2 (1-\cos\theta)\right] \left|\Psi_\Lambda^+\right\rangle - in_3 \sin\theta \left|\Psi_\Lambda^-\right\rangle \right\} \quad (9)$$

(iv) Transformation of $|\Psi^-\rangle$ is as follows:

$$U(\Lambda)\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}} \sum_{\sigma,\sigma'=-1/2}^{1/2} \left\{ \begin{array}{l} \sqrt{\frac{(\Lambda p)^0}{p^0}} D^{(1/2)}_{\sigma,1/2}(R_W(\Lambda,p)) \sqrt{\frac{(\Lambda(-p,p^0))^0}{p^0}} D^{(1/2)}_{\sigma',-1/2}\left(R_W\left(\Lambda,(-p,p^0)\right)\right) |\Lambda p,\sigma\rangle \otimes |-\Lambda p,\sigma'\rangle \\ -\sqrt{\frac{(\Lambda p)^0}{p^0}} D^{(1/2)}_{\sigma,-1/2}(R_W(\Lambda,p)) \sqrt{\frac{(\Lambda(-p,p^0))^0}{p^0}} D^{(1/2)}_{\sigma',1/2}\left(R_W\left(\Lambda,(-p,p^0)\right)\right) |\Lambda p,\sigma\rangle \otimes |-\Lambda p,\sigma'\rangle \end{array} \right\}$$

$$= \frac{1}{\sqrt{2}}\frac{(\Lambda p)^0}{p^0}\left\{\begin{array}{l}(in_1+n_2)\sin\theta\big|\Lambda p,-\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,-\tfrac{1}{2}\big\rangle+(\cos\theta-in_3\sin\theta)\big|\Lambda p,\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,-\tfrac{1}{2}\big\rangle\\-(\cos\theta+in_3\sin\theta)\big|\Lambda p,-\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,\tfrac{1}{2}\big\rangle-(in_1-n_2)\sin\theta\big|\Lambda p,\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,\tfrac{1}{2}\big\rangle\end{array}\right\}$$

$$= \frac{1}{\sqrt{2}}\frac{(\Lambda p)^0}{p^0}\left\{\begin{array}{l}n_2\sin\theta\left(\big|\Lambda p,-\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,-\tfrac{1}{2}\big\rangle+\big|\Lambda p,\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,\tfrac{1}{2}\big\rangle\right)\\+in_1\sin\theta\left(\big|\Lambda p,-\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,-\tfrac{1}{2}\big\rangle-\big|\Lambda p,\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,\tfrac{1}{2}\big\rangle\right)\\-in_3\sin\theta\left(\big|\Lambda p,\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,-\tfrac{1}{2}\big\rangle+\big|\Lambda p,-\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,\tfrac{1}{2}\big\rangle\right)+\\\cos\theta\left(\big|\Lambda p,\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,-\tfrac{1}{2}\big\rangle-\big|\Lambda p,-\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,\tfrac{1}{2}\big\rangle\right)\end{array}\right\}$$

$$= \frac{(\Lambda p)^0}{p^0}\left\{n_2\sin\theta\Big|\Phi_\Lambda^+\Big\rangle-in_1\sin\theta\Big|\Phi_\Lambda^-\Big\rangle-in_3\sin\theta\Big|\Psi_\Lambda^+\Big\rangle+\cos\theta\Big|\Psi_\Lambda^-\Big\rangle\right\} \tag{10}$$

5. The Expectation of the Relativistic Spin Observable in Arbitrarily Boosted Bell States

Given a unit vector$\mathbf{m}$, the projection of spin $\mathbf{S}$ in the direction of $\mathbf{m}$ is obtained as

$$\mathbf{m}.\mathbf{S} = \left[\mathbf{m}.\mathbf{s} + \frac{1}{|\mathbf{p}|^2}\left(\frac{p^0}{M}-1\right)(\mathbf{m}.\mathbf{p})(\mathbf{p}.\mathbf{s})\right] \tag{11}$$

with a magnitude$p^0/2M$. We can accordingly, define normalized spin observable as $\mathbf{S}_m = \frac{\mathbf{m}.\mathbf{s}+\nu(\mathbf{m}.\mathbf{p})(\mathbf{p}.\mathbf{s})}{p^0/2M} = \frac{m.\sigma+\nu(\mathbf{m}.\mathbf{p})(p.\sigma)}{p^0/M}$ where$\nu = \frac{1}{|p|^2}\left(\frac{p^0}{M}-1\right)$. Identifying unit vectors $\mathbf{a}, \mathbf{b}$ in relation to the two particles constituting the paired state, the spin observable for the paired state is obtained as $\mathbf{S}_{ab} = \mathbf{S}_a \otimes \mathbf{S}_b$. Using the abbreviations$A_i = a_i + \nu\,(\mathbf{a}.\mathbf{p})\,p_i$, $B_i = b_i + \nu\,(\mathbf{b}.\mathbf{p})\,p_i$, $i = 1, 2, 3$, we can express the action of this observable on the various basis states as:-

$$S_a\otimes S_b\Big|\Lambda p,\frac{1}{2}\Big\rangle\otimes\Big|-\Lambda p,\frac{1}{2}\Big\rangle = \frac{M^2}{(p^0)^2}\left\{\begin{array}{l}A_3B_3\big|\Lambda p,\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,\tfrac{1}{2}\big\rangle+A_3\left(B_1+iB_2\right)\big|\Lambda p,\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,-\tfrac{1}{2}\big\rangle+\\\left(A_1+iA_2\right)B_3\big|\Lambda p,-\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,\tfrac{1}{2}\big\rangle+\left(A_1+iA_2\right)\left(B_1+iB_2\right)\big|\Lambda p,-\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,-\tfrac{1}{2}\big\rangle\end{array}\right\} \tag{12}$$

$$S_a\otimes S_b\Big|\Lambda p,\frac{1}{2}\Big\rangle\otimes\Big|-\Lambda p,-\frac{1}{2}\Big\rangle = \frac{M^2}{(p^0)^2}\left\{\begin{array}{l}A_3\left(B_1-iB_2\right)\big|\Lambda p,\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,\tfrac{1}{2}\big\rangle-A_3B_3\big|\Lambda p,\tfrac{1}{2}\big\rangle\otimes\\\big|-\Lambda p,-\tfrac{1}{2}\big\rangle+\left(A_1+iA_2\right)\left(B_1-iB_2\right)\big|\Lambda p,-\tfrac{1}{2}\big\rangle\otimes\\\big|-\Lambda p,\tfrac{1}{2}\big\rangle-\left(A_1+iA_2\right)B_3\big|\Lambda p,-\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,-\tfrac{1}{2}\big\rangle\end{array}\right\} \tag{13}$$

$$S_a\otimes S_b\Big|\Lambda p,-\frac{1}{2}\Big\rangle\otimes\Big|-\Lambda p,\frac{1}{2}\Big\rangle = \frac{M^2}{(p^0)^2}\left\{\begin{array}{l}\left(A_1-iA_2\right)B_3\big|\Lambda p,\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,\tfrac{1}{2}\big\rangle+\left(A_1-iA_2\right)\left(B_1+iB_2\right)\\\big|\Lambda p,\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,-\tfrac{1}{2}\big\rangle-A_3B_3\big|\Lambda p,-\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,\tfrac{1}{2}\big\rangle\\-A_3\left(B_1+iB_2\right)\big|\Lambda p,-\tfrac{1}{2}\big\rangle\otimes\big|-\Lambda p,-\tfrac{1}{2}\big\rangle\end{array}\right\} \tag{14}$$

$$S_a \otimes S_b \left|\Lambda p, -\frac{1}{2}\right\rangle \otimes \left|-\Lambda p, -\frac{1}{2}\right\rangle = \frac{M^2}{(p^0)^2} \left\{ \begin{array}{l} (A_1 - iA_2)(B_1 - iB_2)\left|\Lambda p, \frac{1}{2}\right\rangle \otimes \left|-\Lambda p, \frac{1}{2}\right\rangle - (A_1 - iA_2) B_3 \\ \left|\Lambda p, \frac{1}{2}\right\rangle \otimes \left|-\Lambda p, -\frac{1}{2}\right\rangle - A_3 (B_1 - iB_2)\left|\Lambda p, -\frac{1}{2}\right\rangle \otimes \\ \left|-\Lambda p, \frac{1}{2}\right\rangle + A_3 B_3 \left|\Lambda p, -\frac{1}{2}\right\rangle \otimes \left|-\Lambda p, -\frac{1}{2}\right\rangle \end{array} \right\} \tag{15}$$

Furthermore, the expectation values of $S_{ab} = S_a \otimes S_b$ in the various boosted Bell states are given by:

$$\left\langle \Phi_\Lambda^+ \middle| S_{ab} \middle| \Phi_\Lambda^+ \right\rangle = \frac{M^2}{(p^0)^2} \left\{ \begin{array}{l} \left(1 - N_{22}^c - iN_{12}^c\right) \left[\begin{array}{l} A_3 B_3 \left(1 - N_{22}^c + iN_{12}^c\right) - A_3 (B_1 - iB_2)\left(N_2^s - iN_{23}^c\right) + \\ (A_1 - iA_2) B_3 \left(N_2^s + iN_{23}^c\right) + (A_1 - iA_2)(B_1 - iB_2)\left(1 - N_{22}^c - iN_{12}^c\right) \end{array} \right] \\ -\left(N_2^s + iN_{23}^c\right) \left[\begin{array}{l} A_3 (B_1 + iB_2)\left(1 - N_{22}^c + iN_{12}^c\right) + A_3 B_3 \left(N_2^s - iN_{23}^c\right) + \\ (A_1 - iA_2)(B_1 + iB_2)\left(N_2^s + iN_{23}^c\right) - (A_1 - iA_2) B_3 \left(1 - N_{22}^c - iN_{12}^c\right) \end{array} \right] + \\ \left(N_2^s - iN_{23}^c\right) \left[\begin{array}{l} (A_1 + iA_2) B_3 \left(1 - N_{22}^c + iN_{12}^c\right) - (A_1 + iA_2)(B_1 - iB_2)\left(N_2^s - iN_{23}^c\right) - \\ A_3 B_3 \left(N_2^s + iN_{23}^c\right) - A_3 (B_1 - iB_2)\left(1 - N_{22}^c - iN_{12}^c\right) \end{array} \right] + \\ \left(1 - N_{22}^c + iN_{12}^c\right) \left[\begin{array}{l} (A_1 + iA_2)(B_1 + iB_2)\left(1 - N_{22}^c + iN_{12}^c\right) + (A_1 + iA_2) B_3 \left(N_2^s - iN_{23}^c\right) \\ -A_3 (B_1 + iB_2)\left(N_2^s + iN_{23}^c\right) + A_3 B_3 \left(1 - N_{22}^c - iN_{12}^c\right) \end{array} \right] \end{array} \right\}$$

$$= \frac{M^2}{(p^0)^2} \left\{ \begin{array}{l} 2A_3B_3\left[\left(1 - N_{22}^c\right)^2 + \left(N_{12}^c\right)^2\right] + 2\left(A_1B_1 - A_2B_2\right)\left[\left(1 - N_{22}^c\right)^2 - \left(N_{12}^c\right)^2\right] - 4\left(A_1B_2 + A_2B_1\right) \\ \left(1 - N_{22}^c\right)\left(N_{12}^c\right) - 2A_3B_3\left[\left(N_2^s\right)^2 + \left(N_{23}^c\right)^2\right] - 2\left(A_1B_1 + A_2B_2\right)\left[\left(N_2^s\right)^2 - \left(N_{23}^c\right)^2\right] + \\ 4\left(A_1B_2 - A_2B_1\right)\left(N_2^s\right)\left(N_{23}^c\right) - 4A_3\left(1 - N_{22}^c\right)\left[B_1\left(N_2^s\right) - B_2\left(N_{23}^c\right)\right] + 4A_3\left(N_{12}^c\right) \\ \left[B_2\left(N_2^s\right) + B_1\left(N_{23}^c\right)\right] + 4B_3\left(1 - N_{22}^c\right)\left[A_1\left(N_2^s\right) + A_2\left(N_{23}^c\right)\right] - 4B_3\left(N_{12}^c\right)\left[A_2\left(N_2^s\right) - A_1\left(N_{23}^c\right)\right] \end{array} \right\} \tag{16}$$

$$\left\langle \Phi_\Lambda^- \middle| S_{ab} \middle| \Phi_\Lambda^- \right\rangle = \frac{M^2}{(p^0)^2} \left\{ \begin{array}{l} \left(1 - N_{11}^c + iN_{12}^c\right) \left[\begin{array}{l} A_3 B_3 \left(1 - N_{11}^c - iN_{12}^c\right) - A_3 (B_1 - iB_2)\left(iN_1^s + N_{13}^c\right) + \\ (A_1 - iA_2) B_3 \left(iN_1^s - N_{13}^c\right) + (A_1 - iA_2)(B_1 - iB_2)\left(N_{11}^c - 1 - iN_{12}^c\right) \end{array} \right] + \\ \left(iN_1^s - N_{13}^c\right) \left[\begin{array}{l} A_3 (B_1 + iB_2)\left(1 - N_{11}^c - iN_{12}^c\right) + A_3 B_3 \left(iN_1^s + N_{13}^c\right) + \\ (A_1 - iA_2)(B_1 + iB_2)\left(iN_1^s - N_{13}^c\right) - (A_1 - iA_2) B_3 \left(N_{11}^c - 1 - iN_{12}^c\right) \end{array} \right] - \\ \left(iN_1^s + N_{13}^c\right) \left[\begin{array}{l} (A_1 - iA_2) B_3 \left(1 - N_{11}^c - iN_{12}^c\right) - (A_1 + iA_2)(B_1 - iB_2)\left(iN_1^s + N_{13}^c\right) - \\ A_3 B_3 \left(iN_1^s - N_{13}^c\right) + A_3 (B_1 - iB_2)\left(N_{11}^c - 1 - iN_{12}^c\right) \end{array} \right] + \\ \left(N_{11}^c - 1 + iN_{12}^c\right) \left[\begin{array}{l} (A_1 - iA_2)(B_1 + iB_2)\left(1 - N_{11}^c - iN_{12}^c\right) + (A_1 + iA_2) B_3 \left(iN_1^s + N_{13}^c\right) - \\ A_3 (B_1 + iB_2)\left(iN_1^s - N_{13}^c\right) + A_3 B_3 \left(N_{11}^c - 1 - iN_{12}^c\right) \end{array} \right] \end{array} \right\}$$

$$= \frac{M^2}{(p^0)^2} \left\{ \begin{array}{l} 2A_3B_3\left[\left(1 - N_{11}^c\right)^2 + \left(N_{12}^c\right)^2\right] - 2\left(A_1B_1 - A_2B_2\right)\left[\left(1 - N_{11}^c\right)^2 - \left(N_{12}^c\right)^2\right] + 4\left(A_1B_2 + A_2B_1\right) \\ \left(1 - N_{11}^c\right)\left(N_{12}^c\right) - 2A_3B_3\left[\left(N_{13}^c\right)^2 + \left(N_1^s\right)^2\right] + 2\left(A_1B_1 + A_2B_2\right)\left[\left(N_{13}^c\right)^2 - \left(N_1^s\right)^2\right] + \\ 4\left(A_1B_2 - A_2B_1\right)\left(N_{13}^c\right)\left(N_1^s\right) - 4A_3\left(1 - N_{11}^c\right)\left[B_1\left(N_{13}^c\right) + B_2\left(N_1^s\right)\right] - 4A_3\left(N_{12}^c\right) \\ \left[B_2\left(N_{13}^c\right) - B_1\left(N_1^s\right)\right] - 4B_3\left(1 - N_{11}^c\right)\left[A_1\left(N_{13}^c\right) - A_2\left(N_1^s\right)\right] - 4B_3\left(N_{12}^c\right)\left[A_2\left(N_{13}^c\right) + A_1\left(N_1^s\right)\right] \end{array} \right\} \tag{17}$$

$$\left\langle \Psi_\Lambda^+ \middle| S_{ab} \middle| \Psi_\Lambda^+ \right\rangle = \frac{M^2}{(p^0)^2} \left\{ \begin{array}{l} -\left(N_{13}^c - iN_{23}^c\right) \left[\begin{array}{l} -A_3B_3\left(N_{13}^c + iN_{23}^c\right) + A_3\left(B_1 - iB_2\right)\left(1 - N_{33}^c - iN_3^s\right) + \\ \left(A_1 - iA_2\right)B_3\left(1 - N_{33}^c + iN_3^s\right) + \left(A_1 - iA_2\right)\left(B_1 - iB_2\right)\left(N_{13}^c - iN_{23}^c\right) \end{array} \right] + \\ \left(1 - N_{33}^c + iN_3^s\right) \left[\begin{array}{l} -A_3\left(B_1 + iB_2\right)\left(N_{13}^c + iN_{23}^c\right) - A_3B_3\left(1 - N_{33}^c - iN_3^s\right) + \\ \left(A_1 - iA_2\right)\left(B_1 + iB_2\right)\left(1 - N_{33}^c + iN_3^s\right) - \left(A_1 - iA_2\right)B_3\left(N_{13}^c - iN_{23}^c\right) \end{array} \right] + \\ \left(1 - N_{33}^c - iN_3^s\right) \left[\begin{array}{l} -\left(A_1 + iA_2\right)B_3\left(N_{13}^c + iN_{23}^c\right) + \left(A_1 + iA_2\right)\left(B_1 - iB_2\right)\left(1 - N_{33}^c - iN_3^s\right) - \\ A_3B_3\left(1 - N_{33}^c + iN_3^s\right) - A_3\left(B_1 - iB_2\right)\left(N_{13}^c - iN_{23}^c\right) \end{array} \right] + \\ \left(N_{13}^c + iN_{23}^c\right) \left[\begin{array}{l} -\left(A_1 + iA_2\right)\left(B_1 + iB_2\right)\left(N_{13}^c + iN_{23}^c\right) - \left(A_1 + iA_2\right)B_3\left(1 - N_{33}^c - iN_3^s\right) - \\ A_3\left(B_1 + iB_2\right)\left(1 - N_{33}^c + iN_3^s\right) + A_3B_3\left(N_{13}^c - iN_{23}^c\right) \end{array} \right] \end{array} \right\}$$

$$= \frac{M^2}{(p^0)^2} \left\{ \begin{array}{l} 2A_3B_3\left[\left(N_{13}^c\right)^2 + \left(N_{23}^c\right)^2\right] - 2\left(A_1B_1 - A_2B_2\right)\left[\left(N_{13}^c\right)^2 - \left(N_{23}^c\right)^2\right] + 4\left(N_{13}^c\right)\left(N_{23}^c\right)\left(A_1B_2 + A_2B_1\right) \\ -2A_3B_3\left[\left(1 - N_{33}^c\right)^2 + \left(N_3^s\right)^2\right] + 2\left(A_1B_1 + A_2B_2\right)\left[\left(1 - N_{33}^c\right)^2 - \left(N_3^s\right)^2\right] - 4\left(A_1B_2 - A_2B_1\right) \\ \left(1 - N_{33}^c\right)\left(N_3^s\right) - 4A_3\left(1 - N_{33}^c\right)\left[B_1\left(N_{13}^c\right) - B_2\left(N_{23}^c\right)\right] + 4A_3\left(N_3^s\right)\left[B_1\left(N_{23}^c\right) + B_2\left(N_{13}^c\right)\right] \\ -4B_3\left(1 - N_{33}^c\right)\left[A_1\left(N_{13}^c\right) - A_2\left(N_{23}^c\right)\right] - 4B_3\left(N_3^s\right)\left[A_1\left(N_{23}^c\right) + A_2\left(N_{13}^c\right)\right] \end{array} \right\} \tag{18}$$

$$\left\langle \Psi_\Lambda^- \middle| S_{ab} \middle| \Psi_\Lambda^- \right\rangle = \frac{M^2}{(p^0)^2} \left\{ \begin{array}{l} \left(N_2^s + iN_1^s\right) \left[\begin{array}{l} A_3B_3\left(N_2^s - iN_1^s\right) + A_3\left(B_1 - iB_2\right)\left(\cos\theta - iN_3^s\right) - \\ \left(A_1 - iA_2\right)B_3\left(\cos\theta + iN_3^s\right) + \left(A_1 - iA_2\right)\left(B_1 - iB_2\right)\left(N_2^s + iN_1^s\right) \end{array} \right] + \\ \left(\cos\theta + iN_3^s\right) \left[\begin{array}{l} A_3\left(B_1 + iB_2\right)\left(N_2^s - iN_1^s\right) - A_3B_3\left(\cos\theta - iN_3^s\right) - \\ \left(A_1 - iA_2\right)\left(B_1 + iB_2\right)\left(\cos\theta + iN_3^s\right) - \left(A_1 - iA_2\right)B_3\left(N_2^s + iN_1^s\right) \end{array} \right] - \\ \left(\cos\theta - iN_3^s\right) \left[\begin{array}{l} \left(A_1 + iA_2\right)B_3\left(N_2^s - iN_1^s\right) + \left(A_1 + iA_2\right)\left(B_1 - iB_2\right)\left(\cos\theta - iN_3^s\right) + \\ A_3B_3\left(\cos\theta + iN_3^s\right) - A_3\left(B_1 - iB_2\right)\left(N_2^s + iN_1^s\right) \end{array} \right] + \\ \left(N_2^s - iN_1^s\right) \left[\begin{array}{l} \left(A_1 + iA_2\right)\left(B_1 + iB_2\right)\left(N_2^s - iN_1^s\right) - \left(A_1 + iA_2\right)B_3\left(\cos\theta - iN_3^s\right) + \\ A_3\left(B_1 + iB_2\right)\left(\cos\theta + iN_3^s\right) + A_{22}B_{22}\left(N_2^s + iN_1^s\right) \end{array} \right] \end{array} \right\}$$

$$= \frac{M^2}{(p^0)^2} \left\{ \begin{array}{l} 2A_3B_3\left[\left(N_2^s\right)^2 + \left(N_1^s\right)^2\right] + 2\left(A_1B_1 - A_2B_2\right)\left[\left(N_2^s\right)^2 - \left(N_1^s\right)^2\right] + 4\left(N_2^s\right)\left(N_1^s\right) \\ \left(A_1B_2 + A_2B_1\right) - 2A_3B_3\left[\cos^2\theta + \left(N_3^s\right)^2\right] - 2\left(A_1B_1 + A_2B_2\right)\left[\cos^2\theta - \left(N_3^s\right)^2\right] \\ +4\left(A_1B_2 - A_2B_1\right)\left(N_3^s\right)\cos\theta + 4A_3\cos\theta\left[B_1\left(N_2^s\right) + B_2\left(N_1^s\right)\right] + 4A_3\left(N_3^s\right) \\ \left[B_1\left(N_1^s\right) - B_2\left(N_2^s\right)\right] - 4B_3\cos\theta\left[A_1\left(N_2^s\right) + A_2\left(N_1^s\right)\right] + 4B_3\left(N_3^s\right)\left[A_1\left(N_1^s\right) - A_2\left(N_2^s\right)\right] \end{array} \right\} \tag{19}$$

where $N_i^s = n_i \sin\theta$, $N_{ij}^s = n_i n_j \sin\theta$, $N_i^c = n_i\left(1 - \cos\theta\right)$ and $N_{ij}^c = n_i n_j\left(1 - \cos\theta\right)$.

6. Expectation Values of Spin Projections along the axes

Using the above expressions, we can, now, compute the expectation values of the projection of the spin operator $S_{ab} = S_a \otimes S_b$ along the various set of axes. However, the calculations are elaborate and tedious. We can simplify them to some extent without loss of generality by setting the coordinate axes such that the pair momenta is oriented along the x^1 direction i.e. $p \equiv (p, 0, 0)$ so that $n \equiv (0, n_2, n_3)$ and $A\,(B)_i = a\,(b)_i\,(1 + \nu p^2 \delta_{1,i}) = a\,(b)_i \left[1 + \left(\frac{p^0}{M} - 1\right)\delta_{1,i}\right]$. Combinations of unit vectors along the three axes and the corresponding expectation values in the various Bell states are tabulated below:

State	Φ_Λ^+		
$\frac{\mathbf{a}\rightarrow}{\mathbf{b}\downarrow}$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$
$(1,0,0)$	$2\left[(1-N_{22}^c)^2 - (N_2^s)^2 + (N_{23}^c)^2\right]$	$-4\frac{M}{p^0}(N_2^s)(N_{23}^c)$	$-4\frac{M}{p^0}(1-N_{22}^c)(N_2^s)$
$(0,1,0)$	$4\frac{M}{p^0}(N_2^s)(N_{23}^c)$	$-2\frac{M^2}{(p^0)^2}\left[(1-N_{22}^c)^2 + (N_2^s)^2 - (N_{23}^c)^2\right]$	$4\frac{M^2}{(p^0)^2}(1-N_{22}^c)(N_{23}^c)$
$(0,0,1)$	$4\frac{M}{p^0}(1-N_{22}^c)(N_2^s)$	$4\frac{M^2}{(p^0)^2}(1-N_{22}^c)(N_{23}^c)$	$2\frac{M^2}{(p^0)^2}\left[(1-N_{22}^c)^2 + (N_{12}^c)^2 - (N_2^s)^2 - (N_{23}^c)^2\right]$

State	Φ_Λ^-		
$\frac{\mathbf{a}\rightarrow}{\mathbf{b}\downarrow}$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$
$(1,0,0)$	-2	0	0
$(0,1,0)$	0	$2\frac{M^2}{(p^0)^2}$	0
$(0,0,1)$	0	0	$2\frac{M^2}{(p^0)^2}$

State	Ψ_Λ^+		
$\frac{\mathbf{a}\rightarrow}{\mathbf{b}\downarrow}$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$
$(1,0,0)$	$2\left[(1-N_{33}^c)^2 - (N_3^s)^2 + (N_{23}^c)^2\right]$	$4\frac{M}{p^0}(1-N_{33}^c)(N_3^s)$	$4\frac{M}{p^0}(N_3^s)(N_{23}^c)$
$(0,1,0)$	$-4\frac{M}{p^0}(1-N_{33}^c)(N_3^s)$	$2\frac{M^2}{(p^0)^2}\left[(1-N_{33}^c)^2 - (N_3^s)^2 - (N_{23}^c)^2\right]$	$4\frac{M^2}{(p^0)^2}(1-N_{33}^c)(N_{23}^c)$
$(0,0,1)$	$4\frac{M}{p^0}(1-N_{22}^c)(N_2^s)$	$4\frac{M^2}{(p^0)^2}(1-N_{33}^c)(N_{23}^c)$	$2\frac{M^2}{(p^0)^2}\left[(N_{23}^c)^2 - (N_3^s)^2 - (1-N_{33}^c)^2\right]$
State	Ψ_Λ^-		
$\frac{\mathbf{a}\rightarrow}{\mathbf{b}\downarrow}$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$
$(1,0,0)$	$2\left[(N_2^s)^2 + (N_3^s)^2 - \cos^2\theta\right]$	$-4\frac{M}{p^0}(N_3^s)\cos\theta$	$4\frac{M}{p^0}(N_2^s)\cos\theta$
$(0,1,0)$	$4\frac{M}{p^0}(N_3^s)\cos\theta$	$2\frac{M^2}{(p^0)^2}\left[(N_3^s)^2 - (N_2^s)^2 - \cos^2\theta\right]$	$-4\frac{M^2}{(p^0)^2}(N_2^s)(N_3^s)$
$(0,0,1)$	$-4\frac{M}{p^0}(N_2^s)\cos\theta$	$-4\frac{M^2}{(p^0)^2}(N_3^s)(N_2^s)$	$2\frac{M^2}{(p^0)^2}\left[(N_2^s)^2 - (N_3^s)^2 - \cos^2\theta\right]$

The important point to be noted here is that, unlike in the nonrelativistic case, the relativistic case becomes momentum dependent. Furthermore, the composition of the projection axes that lead to maximal violation of Bell's inequalities in the relativistic case is still an open question.

We employ momentum eigenstates in the above formulation as is the case in [6-8]. Mathematically, momentum eigenstates are not square-integrable functions and thus are outside the L2 Hilbert space. Physically, they represent an idealization of the absolutely sharp momentum. Any real state is represented by a wave packet, which is most conveniently decomposed as a (continuous) superposition of the momentum eigenstates. Those superpositions are square-integrable, i.e. normalizable. The convention of using momentum eigenstates is an approximation. It is "strictly true" only as a limiting case. Nevertheless, it is a very good approximation in many cases, e.g. in the scatteringexperiments of a high-energy physics.

In the momentum eigenstates representation, Lorentz boost on a pair state operates through local unitary operators independently on the constituent particle states thereby preserving entanglement and not mixing spin-momentum entanglement so that the spin reduced density matrix is covariant. But when the momenta of the particles is not sharply defined then spin-momentum entanglement as well as spin-spin entanglement becomes frame dependent [15].

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