Statistical Mechanics of Classical N-Particle System of Galaxies in the Expanding Universe

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Abstract: For the distribution of classical non-interacting particles we use Maxwell-Boltzmann's statistics. However, this statistics is not workable for classical interacting particles (galaxies). We attempt to modify the Maxwell-Boltzmann's statistics by incorporating gravitational interaction term in it. The number of ways in which N-particles can have pair interaction due to gravitational interaction is obtained. With the help of entropy maximization we derive the analytical expression for occupation number. Using the modified statistics we obtain the general expressions for different thermodynamical quantities and attempt to derive general distribution function for gravitating particles (galaxies).

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1. Introduction

To understand the distribution of galaxies, it seems natural to explore the role of gravitation. As a cause of large-scale clustering, gravity is a motivating force. The spatial locations of galaxies may be described by many different statistics. These include distribution functions, low-order correlation functions, multi-fractal dimensions, topological genus, minimal spaning trees, percolation and moments. All these statistics, each emphasizes different aspects of distribution, are related (Saslaw 2000). Galaxy distribution function gives information about correlations to all order and about clustering over a very wide range of intenses, sizes and shapes (Sivakoff and Saslaw 2005). As the mutual gravitational interaction of individual galaxies dominate, clustering can be described by quasi-equilibrium thermodynamics (Saslaw 2000; Saslaw and Hamilton 1984; Saslaw

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and Fang 1996) and statistical mechanics (Ahmad et. al 2002,2006; Leong and Saslaw 2004;Saslaw and Ahmad 2008). The results obtained are well consistent with the N-body simulations.

Classical distinguishable particles satisfy Maxwell-Boltzmann's statistics. Indistiguishable quantum particles with half integral spin, obeying exclusion principle, satisfy Fermi-Dirac statistics. Indistiguishable quantum particles with integral spin, obeying no exclusion principle, satisfy Bose-Einstein's statistics. Galaxies are classical gravitating particles which under go quasi-equilibrium evolution in an expanding Universe (Saslaw 2000). Such a system of classical interacting particles (galaxies) satisfy different type of statistics and therefore, aim of the present work is to understand that statistics. In order to develope the statistics of classical interacting particles (galaxies) we have to introduce the gravitational interaction term in the system and this can be done with the help of Virial ratio of potential energy to kinetic energy (Ahmad et. al 2002). The Virial ratio averaged over the entire volume gives us the measure of interaction present in the system. We start with the determination of number of ways in which N-particles can have pair gravitational interaction. With the help of entropy maximization we arrive at the analytical expression of occupation number. The expression for occupation number leads us to different thermodynamical quantities. The present paper is organised as:

In section (2) we obtain the general expression for the statistical weight W_N for the interacting particles. Occupation number $\langle N\rangle$ for interacting particles is obtained in section (3). Thermodynamical quantities are derived in section (4). Finally, the results are discussed in section (5).

2. Probability Distribution Function of Interacting Particles

For a system of non-interacting particles, the number of ways of distributing N-distinguishable particles out of N_o particles in a cell are

$$W_N = \frac{N_o!}{N!(N_o - N)!}. \tag{1}$$

The entropy of the system is $S = \sum \log W_N$ and maximizing it leads to average number of particles $\bar{N}$ associated with energy E_N by using Lagrangian multipliers

$$\bar{N} = e^{-(\alpha+\beta E_N)}, \tag{2}$$

where α and β are Lagrangian multipliers

The probability distribution function $f(N)$ for non-interacting particles (ideal gas) is given by (Landau and Lifshtz 1980)

$$f(N) = \mathbf{c}W_N\left[e^{-(\alpha+\beta E_N)}\right]^N, \tag{3}$$

where **c** is normalization factor and can be determined from the condition $\sum f(N) = 1$ and is $e^{-\bar{N}}$. For $N_o >> N$, $(N_o - N)! \approx N_o!$ and thus giving a distribution function

$$f(N) = \frac{\bar{N}^N}{N!} e^{-\bar{N}}, \tag{4}$$

which is a Poisson distribution.

Now, we extended this method to a system of particles interacting pair-wise gravitationally. For a system with interaction, the particles are dependent i,e. correlated because the particles are no longer free like Ideal gas but are constrained to the forces of mutual interaction. These forces of mutual interaction result in the potential energy of the system and therefore, unlike the perfect gas where potential energy is taken as zero due to absence of any interaction between particles, the total energy of interacting system of particles will include the kinetic energy and potential energy of the particles as well. The inclusion of potential energy term makes the calculation of thermodynamical quantities little bit difficult.
We consider a large system, which consists of large number of cells having the same volume V of radius R_1 (much smaller than the total volume) and average density $\bar{n}$. Both the number of galaxies and their total energy will vary among these cells. In this system, galaxies have a gravitational pairwise interaction.

Our starting point is the Hamiltonian of a system of N particles each of mass m interacting gravitationally with potential energy Φ, having momenta $\mathbf{p}_i$ and average temperature T:

$$E_N = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \Phi(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_N). \tag{5}$$

In general, the gravitational potential energy $\Phi(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_N)$ is function of the relative position vector $\mathbf{r}_{ij} =\mid \mathbf{r}_i - \mathbf{r}_j \mid$ and is the sum of the potential energies of all pairs. Hence

$$\begin{aligned} \Phi(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_N) &= \sum_{1 \le i < j \le N} \Phi(\mathbf{r}_{ij}) \\ &= \sum_{1 \le i < j \le N} \Phi_{ij}(\mathbf{r}). \end{aligned} \tag{6}$$

Here $\Phi_{ij}(\mathbf{r})$ is the potential energy of interaction between ith and jth particles. The interaction potential energy between two galaxies is represented by

$$\Phi_{ij} = -\frac{Gm^2}{\mathbf{r}_{ij}}. \tag{7}$$

Since, in the absence of any interaction in the system the total energy will be only kinetic energy K. As soon as interaction gets introduced in the system correlation potential energy increases and kinetic energy decreases. Therefore, the size of correlation may be defined as

$$\frac{E_N}{K} = \left(1 + \frac{\Phi(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_N)}{\sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m}}\right) \tag{8}$$

One thing is obvious that for one particle correlation has no meaning. Therefore, probability of getting N particles interacting pair-wise in the correlated cell is $\left(1 + \Phi(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_N)/\sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m}\right)^{N-1}$. With the help of equation (1), the total number ways of distributing N particles in a correlated cell of volume V is

$$W_N = \frac{N_o!}{N!(N_o - N)!}\left[1 + \frac{\Phi(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_N)}{\sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m}}\right]^{N-1}. \tag{9}$$

The ratio of potential energy to the kinetic energy can be approximated with virial ratio of potential energy and kinetic energy averaged over the entire region i,e. $\langle Gm^2/rT \rangle$ (with energy units in which Boltzmann's constant k=1) (Ahmad et. al 2002). Thus equation (9) can be written as:

$$W_N = \frac{N_o!}{N!(N_o - N)!}\left(1 + \langle Gm^2/rT \rangle\right)^{N-1}, \tag{10}$$

$$W_N = \frac{N_o!}{N!(N_o - N)!}(1 + \langle x \rangle)^{N-1}, \tag{11}$$

with $\langle x \rangle = Gm^2/\langle r \rangle T$ is virial ratio. The factor $\langle x \rangle$ can be calculated from

$$\langle x \rangle = \frac{\int_{R_1} x dV}{\int_{R_1} dV} = \frac{\int_{R_1} x dV}{V}, \tag{12}$$

with R_1 as the radius of the cell. The factor $\langle x \rangle$ is different for point and non-point masses.

3. Occupation Number for Interacting Particles

Once the statistical weight is calculated we can determine the occupation number, which will determine deviation of interacting particles from an ideal gas. The entropy for such system is

$$S = \sum \log W_N, \tag{13}$$

$$S = \sum \log \frac{N_o!}{N!(N_o - N)!}(1 + \langle x \rangle)^{N-1}. \tag{14}$$

Equation (14) gives the entropy of the system in which particles are interacting in pair-wise manner. In our case, the interaction in the system is gravitational. As gravitational interaction is pair wise and additive, the number of particles should be more than unity $N >> 1$. Therefore $N - 1$ can be approximated with N. Moreover, for the entire system the total number of particles N_o is very very large as compared to N then $N_o - N \approx N_o$. Taking these things into consideration and using Stirling's approximation, equation (14) after differentiation can be written as

$$\partial S = \sum \left[-\log N \partial N + \log\left(1 + \langle x \rangle\right) \partial N + \frac{N \frac{\partial \langle x \rangle}{\partial N} \partial N}{(1 + \langle x \rangle)} \right]. \tag{15}$$

Here, it is worth to mention that $\langle x \rangle$ is function of N also, that is why we have the term $\partial \langle x \rangle / \partial N$.
The term $\partial S = 0$ is the condition for the maximization of entropy and obviously corresponds to the most probable state. With the help of this condition we get

$$\sum \left[\log N \partial N - \log\left(1 + \langle x \rangle\right) \partial N - \frac{N \frac{\partial \langle x \rangle}{\partial N} \partial N}{(1 + \langle x \rangle)} \right] = 0. \tag{16}$$

Using Lagrangian multipliers, equation (16) gives the occupation number as

$$\langle N \rangle = \left(1 + \langle x \rangle\right) e^{\frac{N \frac{\partial \langle x \rangle}{\partial N}}{(1 + \langle x \rangle)}} e^{-(\alpha + \beta E_N)}. \tag{17}$$

From equation (17) it is clear that the occupation number for interacting particles is different from that of occupation number for non-interacting particles. Occupation number for interacting system depends upon the averaged virial ratio. More its value is more the number of particles will be in the energy shell E_N. For ideal gas system the above occupation number, given by equation (17), reduces to occupation number given by equation(2)

4. Thermodynamical Quantities

As our entire system is very large and is comprised of many cells of volume V. We can suppose that the large system is possessing a spherical volume V_o with radius R_o. Then the total number of particles N_o in the large system can be given as

$$N_o = \int_0^{R_o} d\langle N \rangle. \tag{18}$$

The energy term E_N is function of position co-ordinate **r** and momentum co-ordinate **p** of galaxies within the volume, therefore, equation (18) gives

$$N_o = \int_0^{R_o} (1 + \langle x \rangle) e^{\frac{N \frac{\partial \langle x \rangle}{\partial N}}{(1 + \langle x \rangle)}} e^{-(\alpha + \beta E_N)} d^3\mathbf{p} d^3\mathbf{r}. \tag{19}$$

Here, it may be noted that entire system (large system) is not correlated, only a cell of volume V is in correlated state. therefore (19) can be written as (after making it dimensionless) after integrating

$$N_o = (1 + \langle x \rangle) e^{\frac{N \frac{\partial \langle x \rangle}{\partial N}}{(1+\langle x \rangle)}} e^{-\alpha} \left(\frac{2\pi m T}{h^2} \right)^{3/2} V_o. \tag{20}$$

Equation (18) further gives

$$e^{-\alpha} = \frac{N_o}{V_o (1 + \langle x \rangle)} e^{-\frac{N \frac{\partial \langle x \rangle}{\partial N}}{(1+\langle x \rangle)}} \left(\frac{2\pi m T}{h^2} \right)^{-3/2}. \tag{21}$$

For the uniform distribution of matter in the Universe $N_o/V_o = N/V$ and using $\alpha = -\mu/T$, where μ is chemical potential of the system which measures the exchange of particles, equation (21) can be written as

$$e^{\frac{\mu}{T}} = \frac{N}{V (1 + \langle x \rangle)} e^{-\frac{N \frac{\partial \langle x \rangle}{\partial N}}{(1+\langle x \rangle)}} \left(\frac{2\pi m T}{h^2} \right)^{-3/2}. \tag{22}$$

On solving equation (22) further, we get

$$\frac{\mu}{T} = \log \left(\frac{N}{V} T^{-3/2} \right) + \log \left[\frac{1}{(1 + \langle x \rangle)} \right] - \frac{N \frac{\partial \langle x \rangle}{\partial N}}{(1 + \langle x \rangle)} + \log \left(\frac{2\pi m}{h^2} \right)^{-3/2}. \tag{23}$$

This is general form of chemical potential obtained for gravitating system in which particles are interacting pair-wise. Once we get form of μ we can easily evaluate other thermodynamical quantities, for example, from the thermodynamical definition of μ, we have

$$F = \int \mu dN, \tag{24}$$

where F is the free energy. We can also find entropy S and total internal energy U as

$$S = -\left(\frac{\partial F}{\partial T} \right)_{V,N}, \tag{25}$$

and

$$U = F + TS. \tag{26}$$

All these thermodynamical quantities when evaluated are found to have same form as obtained earlier (Ahmad et. al 2002). The determination of (μ/T) is the basic parameter which can be used for evaluation of distribution function $f(N)$ of galaxies in a system.The full distribution function function $f(N)$ is given by equation (3), however, with W_N given by equation (11) (assuming that the total number of particles N_o is very large as compared to number of particles N in volume V such that $(N_o - N)! \approx N_0!$) and $e^{-N(\alpha+\beta E_N)}$ by equation (17) such that

$$f(N) = \mathbf{c} \frac{1}{N!} \frac{\langle N \rangle^N}{(1 + \langle x \rangle)} e^{-N^2 \frac{\partial \langle x \rangle}{\partial N}}. \tag{27}$$

The constant **c** in equation (27) can be calculated from the condition of normalization $\sum f(N) = 1$. If solved for point particles, the distribution function given by equation (27) may yield the form already obtained (Saslaw and Hamilton 1984; Ahmad et. al 2002)

$$f(N) = \frac{\bar{N}(1-b)}{N!}\left[\bar{N}(1-b) + Nb\right]^{N-1} e^{-\left[\bar{N}(1-b)+Nb\right]}, \quad (28)$$

with $b = b_o\bar{n}T^{-3}/(1 + b_o\bar{n}T^{-3})$.

5. Discussion

Galaxies are classical distinguishable particles. However, there is one main difference which does not make them Maxwell- Boltzmann's particles i.e. they are interacting pair-wise. Maxwell- Boltzmann's particles, though distinguishable do not interact, however, in our case galaxies are interacting gravitationally. If galaxies are non interacting we can use Maxwell- Boltzmann's statistics but the presence of gravitational interaction in the system of galaxies makes their statistics complicated. An attempt is made to develop the statistics of the gravitating systems of galaxies by introducing the interaction via virial ratio of potential energy to the kinetic energy of the galaxies. The first impact of interaction is seen on the statistical weight which is the number of ways of distributing particles in a correlated cell. As gravitational interaction is pair wise and additive in nature, for a single particle correlation has no meaning that is why the power $(N-1)$ appears on the correction term in equation (9). The statistical weight obtained for interacting systems reduces back to the statistical weight of non-interacting system for $\langle x\rangle = 0$. The entropy maximization condition $\partial S = 0$, yields the occupation number $\langle N\rangle$ which is modified form of Maxwell- Boltzmann's distribution for interacting particles. Very interesting fact about occupation number is that for $\langle x\rangle \to \infty$ it tends to infinity.

Moreover, the full distribution function is obtained by normalizing condition and is modified as compared to the Poisson distribution function. Further, the thermodynamical quantities are modified for the gravitating system of the galaxies.

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